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Physics 8.04 October 11, 1995

# Quantum Physics I, Spring 1995

## Lecture Notes on Compton Scattering

## 1 A Working Hypothesis from the Correspondence Principle: $p_{photon} = h\nu/c$.

The hypothesis which the Compton Scattering experiment most clearly highlights is the idea that individual photons from light of frequency $\nu$ should carry a momentum $p = E/c = h\nu/c$, where $h$ is Planck's constant, $6.63 \times 10^{-27} erg\, s$. This hypothesis arises naturally as a consequence of the correspondence principle and the observation that, *as a direct consequence of Maxwell's equations*, a classical electromagnetic wave of energy $U$ carries a momentum $U/c$.

We can thus perform a thought-experiment where a monochromatic beam of light of frequency $\nu$ and energy $U$ is directed at an absorbing plate. With our faith in Maxwell's equations, we know that after the plate absorbs the radiation, it will recoil in the experiment with a momentum given by $P = U/c$. On the other hand, from the Planck Radiation formula and the photoelectric effect, we know that microscopically the energy of the beam arrived at the plate in a series of $N = U/h\nu$ discrete packets or *photons*. To explain the recoil of the plate from this microscopic picture, it is natural to associate with each of the photons a momentum as well as an energy. To explain the magnitude of the observed recoil, the average momentum carried by the photons of must be the total recoil momentum divided by the number of photons, $< p >= P/N = (U/c)/(U/h\nu) = h\nu/c$. Since all photons of frequency $\nu$ carry precisely the same energy, a further natural working hypothesis would be that they all carry the same momentum as well so that $p = h\nu/c = h/\lambda$. Note that this cannot be concluded strictly from the though experiment, however, as we shall see this hypothesis is completely consistent with the experimental observations.

## 2 Consequence of the Hypothesis

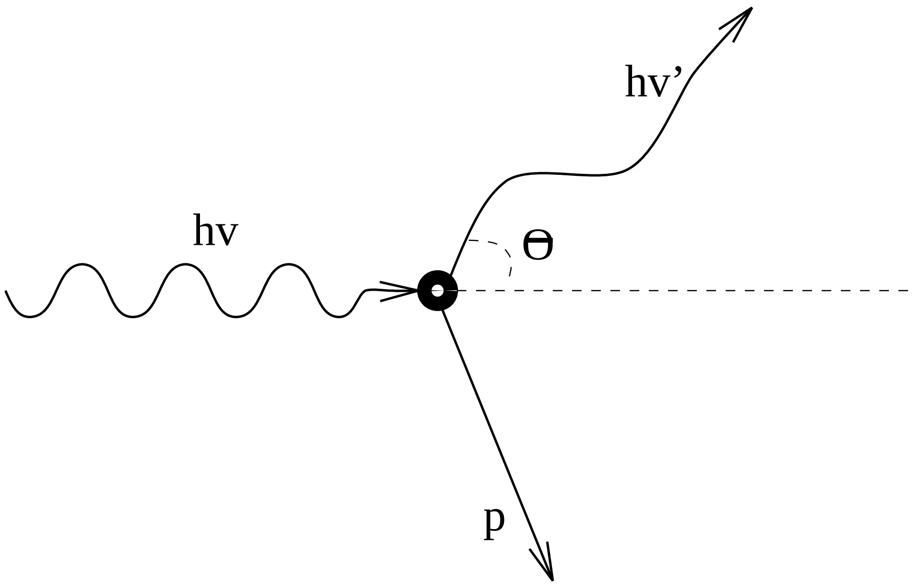
The most natural test of this hypothesis is to analyze a collision event where we assume that the photon carries a momentum $h\nu/c$ and to see whether our expectations are in line with the experimental results. The collision event which we shall analyze is diagrammed in Figure 2.

Here we have an incoming photon of frequency $h\nu$ colliding with a *stationary* particle of mass m (in the original experiment, the particles were electrons). After the collision, the photon transfers some of its momentum to the particle which recoils with a momentum $p$. With its momentum reduced, the photon then emerges with a new, lower frequency $\nu'$.

Note that this effect is difficult to understand within the classical picture given by Maxwell's equations. Classically, we would expect the response of the electron to an incoming electromagnetic wave of frequency $\nu$ to be an oscillatory motion of frequency $\nu$. With the electron oscillating at this frequency, we would then expect it to radiate electromagnetic energy in all directions but with the same frequency as its motion, $\nu$. The shift in frequency we are about to predict is a purely quantum effect arising from the discrete nature of light.

To analyze the collision, we will use conservation of momentum and energy. To keep our results general, we shall anticipate the possibility of using very high energy photons that may leave the particle recoiling at relativistic speeds. We will use the relativistic form for the energy-momentum relationship of a particle,

$$E(p) = \sqrt{(mc^2)^2 + (cp)^2}.$$




Note that when the particle is at rest ($p = 0$) we have $E = mc^2$, the rest mass energy of the particle. Further, when the electron has relatively little momentum

$$E(p) = \sqrt{(mc^2)^2 + (cp)^2} = mc^2\sqrt{1 + (p/mc)^2} \simeq mc^2\left(1 + \frac{1}{2}(p/mc)^2\right) = mc^2 + (p/mc)^2$$

which is just the rest mass energy plus the usual non-relativistic form for the kinetic energy.

For energy to be conserved in our collision, the sum of the initial and final energies must be equal,

$$h\nu + mc^2 = h\nu' + \sqrt{(mc^2)^2 + (cp)^2}.$$

(Recall that we suppose are particle to be at rest initially.) Because we will soon apply conservation of momentum, it proves convenient to solve this equation for the momentum of the electron,

$$\begin{aligned}
h\nu + mc^2 &= h\nu' + \sqrt{(mc^2)^2 + (cp)^2} \\
\sqrt{(mc^2)^2 + (cp)^2} &= h\nu - h\nu' + mc^2 \\
(mc^2)^2 + (cp)^2 &= (h\nu - h\nu' + mc^2)^2 \\
(mc^2)^2 + (cp)^2 &= (h\nu)^2 + (h\nu')^2 + (mc^2)^2 - 2(h\nu)(h\nu') + 2(mc^2)(h\nu - h\nu') \\
(cp)^2 &= (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu') + 2(mc^2)(h\nu - h\nu') \qquad (1)
\end{aligned}$$

To apply conservation of momentum, we rearrange the momentum vector diagram as in Figure 2.

We now apply the law of cosines,

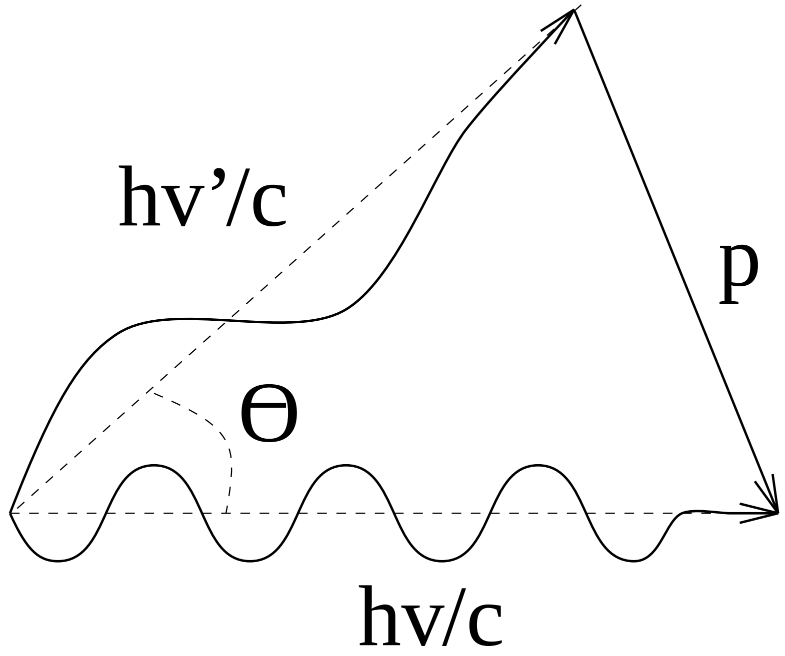
$$(p)^2 = (h\nu/c)^2 + (h\nu'/c)^2 - 2(h\nu/c)(h\nu'/c)\cos\theta.$$

To ease combination with Equation (1), we first multiply both sides by $c^2$,

$$(cp)^2 = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cos\theta. \qquad (2)$$

Now, we may set the right hand sides of (1) and (2) equal,

$$\begin{aligned}
(h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu') + 2(mc^2)(h\nu - h\nu') &= (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cos\theta \\
-2(h\nu)(h\nu') + 2(mc^2)(h\nu - h\nu') &= -2(h\nu)(h\nu')\cos\theta \\
(mc^2)(h\nu - h\nu') &= (h\nu)(h\nu')(1 - \cos\theta)
\end{aligned}$$




Dividing both sides by $(h\nu)(h\nu')$ and using the wavelength frequency relationship $c = \nu\lambda$ we get our final result,

$$
\begin{aligned}
(mc^2)(\frac{1}{h\nu'} - \frac{1}{h\nu}) &= (1 - \cos\theta) \\
(mc^2)(\frac{\lambda'}{hc} - \frac{\lambda}{hc}) &= (1 - \cos\theta) \\
\frac{mc}{h}(\lambda' - \lambda) &= (1 - \cos\theta) \\
(\lambda' - \lambda) &= \left(\frac{h}{mc}\right)(1 - \cos\theta) \\
\Delta\lambda &\equiv \lambda_0(1 - \cos\theta), \qquad (3)
\end{aligned}
$$

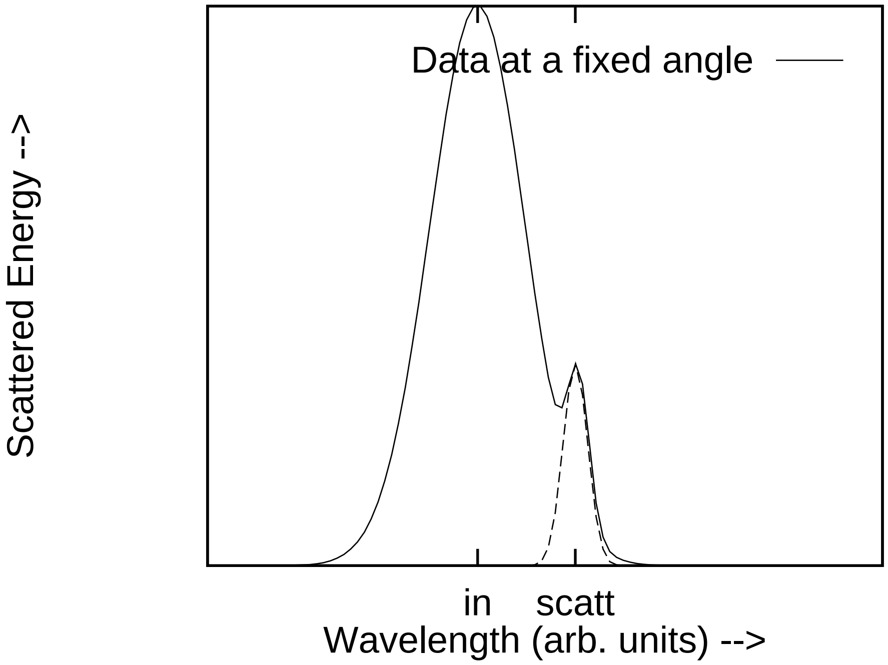
where we have defined the quantity $\lambda_0 = \left(\frac{h}{mc}\right)$, which has the dimensions of length and is known as the Compton wavelength. For the electron it has the numerical value $\lambda_0 \simeq 0.0243$Å.

## 3 Confirmation in the Experiments

Note that Equation (3) predicts a shift in the wavelength and frequency of the re-radiated light, an effect unexpected in classical physics. Note also the the magnitude of the shift in wavelength $\Delta\lambda = \frac{h}{mc}(1 - \cos\theta)$ is predicted to be *independent* of the frequency of the incoming radiation. It has the same value whether visible light, $X-$rays or $\gamma$-rays are used. When using visible light $\lambda \simeq 5100$Å the shift is hardly noticeable, but as one moves to the higher frequency and shorter wavelengths of $X-$ and $\gamma-$ rays, the shift becomes quite significant.

These predictions are borne out the in experiments. Typically, one sends $\gamma-$rays into a solid target. Because the $\gamma-$rays have energies in excess of $1MeV$, the binding of the electrons to the solid is relatively insignificant (at most hundreds of $keV$ for the most tightly bound electrons in the heaviest nuclei) and we can ignore the electrons' motion within the solid. Typical experimental results might appear as in Figure 3.

Experimentally, one finds a peak centered at the incoming wavelength ("in" in the figure) and an additional peak shifted in exact accord with (3) ("scatt" in the figure). The peak at the incoming wavelength is easy to understand from our result (3). The photons in the experiment scatter off of not only the electrons but also the nuclei in the solid. In addition it is possible that when the photons hit a particularly tightly bound electron, the recoil is taken up by the atom a whole and not by the


Data at a fixed angle
Scattered Energy -->
in
scatt
Wavelength (arb. units) -->

electron alone. Finally, events are also possible where the *entire crystal* recoils. In these cases, one must use the mass of the recoiling object in the formula for the Compton wavelength, $\lambda_0 = \frac{h}{mc}$. As the mass of the nuclei and atoms in the target are generally at least 40,000 times more massive than the electron, the shift from these alternate scattering events is puny in comparison to the shift from the electrons and does not show up in the experimental results.

# 4 Comments

## 4.1 The Classical Limit

Note that the shift we predict is a purely quantum effect. In accordance with the concept of the classical limit, we find that as $h \longrightarrow 0$, $\lambda_0 \longrightarrow 0$, and we recover the classical prediction of no frequency shift in the scattered radiation. We interpret this either as the statement that there is no frequency shift as we turn quantum mechanics "off" ($\hbar \longrightarrow 0$), or as recognition of the fact that if we were to use a macroscopic values for the mass in $\lambda_0 = \frac{h}{mc}$, we would find an extremely tiny shift that would not be noticeable in experiments performed on the macroscopic objects of the type which concern the experiments leading to the classical Maxwell's equations.

## 4.2 Impossibily of electrons absorbing photons in free space

Finally note that when the wavelength of the incoming light $\lambda$ exceeds $\lambda_0$, there is *always* an outgoing photon carrying a finite amount of energy $h\nu'$. It is impossible for an electron in free space to completely absorb the energy and momentum of a photon and still satisfy the basic laws of the conservation of energy and momentum. You will recall, however, that this complete absorption is precisely what we assumed happened in our analysis of the photoelectric effect. The resolution of this "paradox" is just to observe that the electrons in the photoelectric effect are not free but are inside of a crystal which can help absorb the momentum of the incoming photons.