Massachusetts Institute of Technology

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Lecture Notes for de Broglie Waves

1 The Experimental Observation

From the Davisson-Germer, G.P. Thomson and other experiments, we learn *empirically* that the particles detected in experiments exhibit wave-like diffraction and interference effects with a wavelength λ given by

$$\lambda = \frac{h}{p}, \tag{1}$$

where p $(= \sqrt{2mE})$ is the magnitude of the momentum of the particles in the experiment and h is a constant with the experimental value $h = 6.63 \times 10^{-27} erg\, s$, precisely the same Planck's constant from the black body radiation formula. For the moment we leave the fact that this constant is the same for all particles, including photons, as an experimental fact.

2 The Correspondence Principle

We can not ignore the the wide range of physical phenomena which classical physics describes well. The correspondence principle emphasizes these successes. The *correspondence principle* is the recognition that any valid theory must be able to describe the phenomena of classical physics which we experience every day. Using this simple principle, one can impose surprisingly strong constraints and conditions on our new, more fundamental, quantum theory. This procedure ensures that quantum theory is consistent with the empirical observations of classical physics.

The particular phenomena with which we shall impose consistency in this set of notes is the fact that, as J.J. Thomson's observed, the electrons in cathode-ray tube experiments follow trajectories governed by Newton's law, $F = ma$. The fact that electrons follow Newtonian trajectories must be reconciled with the fact that they exhibit wave behavior with a wavelength given by (1) above.

3 Wave Packets

The notion of a particle following a Newtonian trajectory involves the concept of of a localized object following a well defined path in space. This notion is not unfamiliar in the theory of waves. It *corresponds* to the wave theory notion of a wave packet. In wave theory the superposition of a packet or group of waves waves with slightly different wavelengths can produce a wave which is localized in space. With an appropriate combination, the center of the group of waves will travel with an average overall velocity known as the *group velocity.* The two descriptions can be reconciled and will correspond provided that the group velocity c_g in the wave description and the particle velocity v along the Newtonian trajectory description can be made to *always* agree!

As an example of a wave packet, consider the combination of two waves

$$\Psi_1 = e^{i(\frac{2\pi}{\lambda_1}x - 2\pi\nu_1 t)}$$

and

$$\Psi_2 = e^{i(\frac{2\pi}{\lambda_2}x - 2\pi\nu_2 t)}.$$

Defining the phases

$$\phi_{1,2} \equiv \frac{2\pi}{\lambda_{1,2}} x - 2\pi\nu_{1,2} t$$

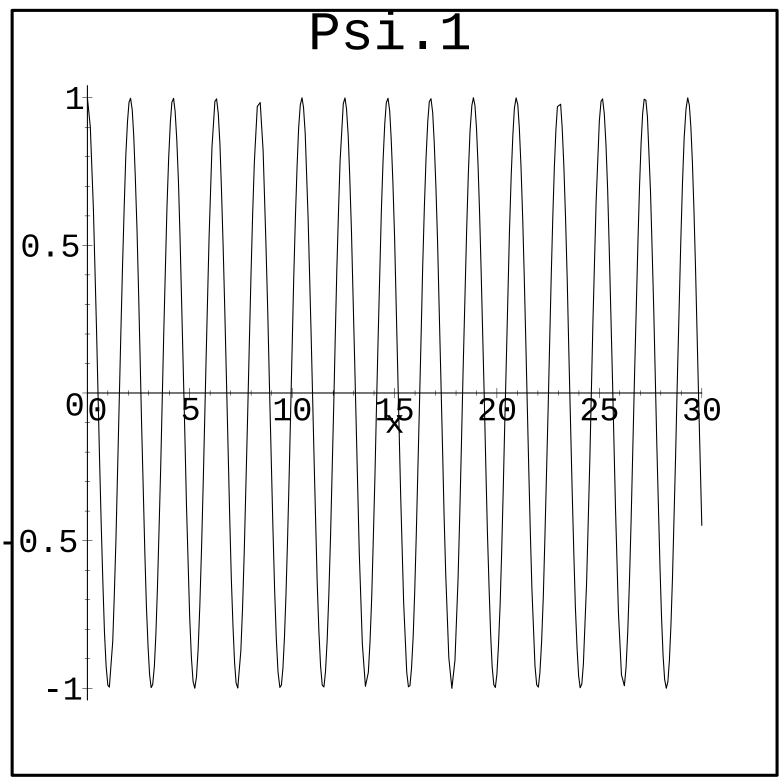
and the phase difference

$$\Delta \equiv \phi_2 - \phi_1 = 2\pi \left((\frac{1}{\lambda_2} - \frac{1}{\lambda_1}) x - (\nu_2 - \nu_1) t \right) = 2\pi \left((\Delta \frac{1}{\lambda}) x - (\Delta\nu)\, t \right),$$

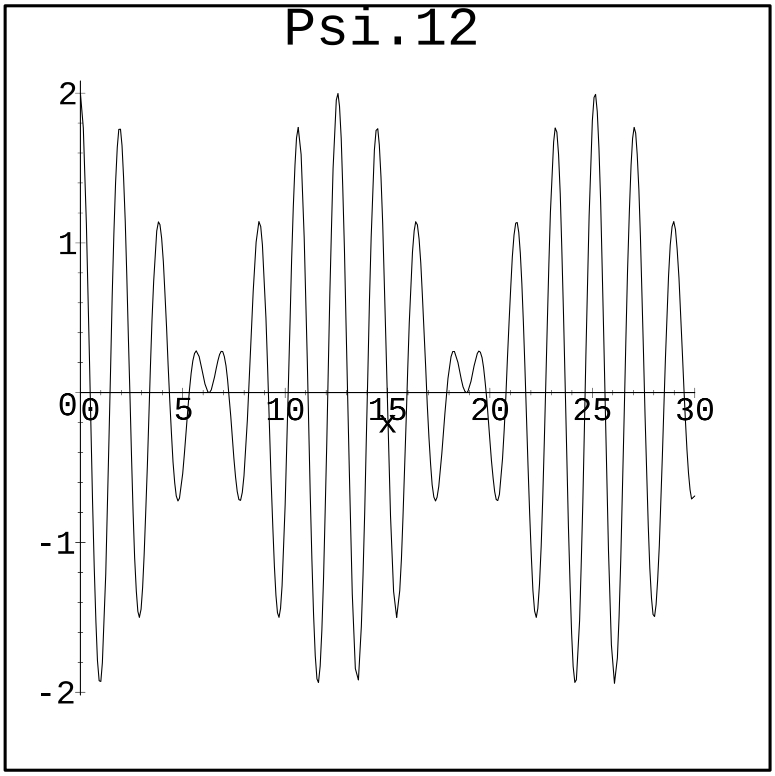
we analyze the superposition of these waves as

$$\begin{aligned} \Psi_{12} \equiv \Psi_1 + \Psi_2 = & \quad e^{i\phi_1} + e^{i\phi_2} \\ & = e^{i\phi_1}(1 + e^{i\Delta}) \\ & = e^{i\phi_1}(Ae^{i\theta}), \\ & = Ae^{i\phi_1 + \theta}, \qquad (2) \\ & \qquad (3) \end{aligned}$$

where $|A|^2 \equiv |1 + e^{i\Delta}|^2 = 2(1 + \cos\Delta)$ and $\tan\theta = \sin\Delta/(1 + \cos\Delta)$. If the waves Ψ_1 and Ψ_2 are close in frequency, we see that the new wave Ψ_{12} looks much like the original waves $e^{i\phi_1} \sim e^{i\phi_2}$, but for a phase shift θ and a modulation in amplitude A. (See figures below.)



Real Part of Ψ_1



REAL PART OF $\Psi_{12} \equiv \Psi_1 + \Psi_2$

Note that the modulation in amplitude $A = \sqrt{2(1+\cos(\Delta))} = \sqrt{2(1+\cos 2\pi(\Delta\frac{1}{\lambda}\,x - \Delta\nu\,t))}$, tends to localize the wave group Ψ_{12} into regions which travel at a velocity c_g given by $\Delta\frac{1}{\lambda}\,x - \Delta\nu\,t =$ const $\Rightarrow c_g \equiv \frac{\Delta x}{\Delta t} = \frac{\Delta\nu}{\Delta\frac{1}{\lambda}}$. We did not achieve perfect localization with this wave "group" of two waves because we used so few waves. In general, the more waves that make up the group, the better it can be localized. When dealing with waves, there is generally a relationship between the frequency and wavelength of the waves $\nu = f(\frac{1}{\lambda})$. This is known as the dispersion relationship and is a a basic property of each type of wave we encounter. Although here we only used two waves in our group, one can give a similar argument in the general case to show that even when combining many waves, as long as their frequencies and wavelengths fall in a narrow interval so that $\Delta\nu, \Delta\frac{1}{\lambda} \longrightarrow 0$, then

$$c_g = \frac{\partial\nu}{\partial\frac{1}{\lambda}}, \tag{4}$$

which you may also see sometime written

$$c_g = \frac{\partial\omega}{\partial k},$$

where $\omega \equiv 2\pi\nu$ and $k \equiv 2\pi/\lambda$.

4 Hamilton's equation

To achieve correspondence between the two descriptions, we would now like to insist that the group velocity given by $c_g = \frac{\partial \nu}{\partial \frac{1}{\lambda}}$ equal the Newtonian particle velocity v. ($c_g \longleftrightarrow v$.) We also have from the experiments the empirical observation (1) that $\frac{1}{\lambda} \longleftrightarrow p/h$. Substituting these correspondences in (4) leaves

$$v = \frac{\partial \nu}{\partial p/h} = \frac{\partial h\nu}{\partial p} = \frac{\partial [?]}{\partial p}, \tag{5}$$

which relates the velocity of a particle on a Newtonian trajectory to the derivative of some as yet unspecified Newtonian quantity ([?]) with respect to the particle's momentum. To complete the correspondence, we must identify a derivative relationship like this from classical Newtonian mechanics. Such a relation is well known in advanced mechanics and is known as one of the two canonical equations of motion of Hamilton.

To derive the relationship we need, we use only the basic concepts of the conservation of momentum and of energy. Conservation of momentum tells us that if we apply an external force F^{ext} to a particle in motion along a trajectory, then the momentum of the particle must change according to

$$\frac{d}{dt} p(t) = F^{ext}. \tag{6}$$

As the momentum of the particle changes, so will its energy. In this derivation, it is useful for us to write the energy of a particle as a function of its momentum, rather than the more familiar procedure of writing the energy as a function of the velocity of the particle. This form is generally more useful because it relates two conserved quantities. Writing the energy this way is so useful, in fact, that this form is given a special name in honor of Hamilton. It is called the "Hamiltonian" and by convention is written with the letter H, $E = H(p)$. In this argument, we will leave $H(p)$ arbitrary. It may take the usual form used in classical mechanics,

$$H(p) = \frac{p^2}{2m} \left(= \frac{1}{2} m v^2 \right), \tag{7}$$

or the relativistic form we used when studying Compton scattering

$$H(p) = \sqrt{(mc^2)^2 + (cp)^2}, \tag{8}$$

or the energy-momentum relationship for a photon

$$H(p) = cp, \tag{9}$$

or some other mysterious form we have not yet encountered.

With the energy written in this special way, the energy of our particle will vary with the momentum $p(t)$ through the relationship $E = H(p(t))$. The energy of the particle thus changes at the rate

$$\frac{dE}{dt} = \frac{\partial H(p)}{\partial p} \frac{dp(t)}{dt} = \frac{\partial H(p)}{\partial p} F^{ext},$$

where we have used Eq. (6) to relate the force and the rate of change of the momentum. Applying the law of the conservation of energy, this rate of change of energy of the particle along its trajectory must also must be the rate at which the external force F^{ext} does work on the particle, $\frac{dW}{dt} = F^{ext} v$. We thus have

$$F^{ext} v = \frac{dW}{dt} = \frac{dE}{dt} = \frac{\partial H(p)}{\partial p} F^{ext} \tag{10}$$

As so we conclude that, in general

$$v = \frac{\partial H}{\partial p} \left(= \frac{\partial E}{\partial p} \right). \tag{11}$$

This is easily verified in the cases above. For the classical particle (7),

$$\frac{\partial \frac{p^2}{2m}}{\partial p} = \frac{p}{m} = v.$$

For the photon (9)

$$\frac{\partial cp}{\partial p} = c.$$

The relativistic case (8) involves a bit more work, but the relation

$$\frac{\partial}{\partial p}\sqrt{(mc^2)^2 + (cp)^2} = \frac{c(cp)}{\sqrt{(mc^2)^2 + (cp)^2}} = v$$

may be solved for p in terms of v to give the more familiar

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

5 Completing the correspondence

We now have both the group velocity c_g and the Newtonian velocity v in terms of partial derivatives and so can complete the correspondence. Comparing (5) and (11) we must have

$$E \longleftrightarrow h\nu,$$

which is the second de Broglie relation mapping energies to frequencies. Gathering the two de Broglie relations together,

$$\begin{aligned} E &\longleftrightarrow h\nu \\ p &\longleftrightarrow h/\lambda \end{aligned} \tag{12}$$

6 Discussion

Note that it is a quite natural consequence of the correspondence principle that precisely the same constant appears in both the momentum-wavelength and energy-frequency de Broglie relations (12). Although these relations related two different pairs of physical quantities, only one, and not two, fundamental constants are involved. Ultimately, the equality of these constants results from our definition of *work* as *force times distance.* If physicists had happened to decide to define work as *two* times the value of force times distance (so that we could write the kinetic energy of a particle as mv^2 instead of $\frac{1}{2}mv^2$), then the two constants appearing in the de Broglie relations *would be* different, *but by* precisely the same factor of *two.*

It is also interesting to ponder the question of why the same value of Planck's constant should apply for photons as well as for every single kind of particle. This also comes about as a consequence of conservation laws. To link the value of Planck's constant for two different types of particles, we must consider a situation where two unlike particles interact as for example in the electron-photon collision in Compton Scattering. A full discussion of how to describe with waves a system containing more than one particle is beyond the scope of the course at this stage. We can, however, give a plausibility argument for the one fact which we need from a more advanced multiple particle description. To combine the energy of two particles (say when they are far enough apart that they are no longer interacting), we know that we simply *add* the energies of the two separate particles. As we have just seen, energy and frequency are related in direct proportion. So, it is *plausible* that when combining the waves describing these two particles the frequencies will add as well. If we accept this idea as plausible, we see that for energy $E_1 + E_2 = h_1\nu_1 + h_2\nu_2$ and frequency $\nu_1 + \nu_1$ *both to be conserved simultaneously* in all collisions exchanging energy/frequency between the two particles, the particles must share the same value for Planck's constant $h_1 = h_2 = h$.