Massachusetts Institute of Technology

Department of Physics

Physics 8.04 October 12, 1995

Quantum Physics I, Spring 1995

As in the syllabus, "FT"=*French and Taylor*, "ER"=*Eisberg and Resnick*, "FLP"=*Feynman Lectures, vol. III*, and the numbers refer to sections. Here the numbers listed are the pre-lecture readings. The "extended" readings are also given, but within parenthesis. Reading assignments:

(**Last Week**) Tue. March 7, Lecture 8 : FT: 1.7-1.9
Tue. March 14, Lecture 9: FT: "Linear Momentum Operators," p. 443-444; ER: 5.4
Thurs. March 16, Lecture 10: FT 3.1-3.4

Problem Set IV

(Due Friday, March 17 at 5:00 pm sharp.)

Work requested of the students is highlighted by either a question mark or underlining.

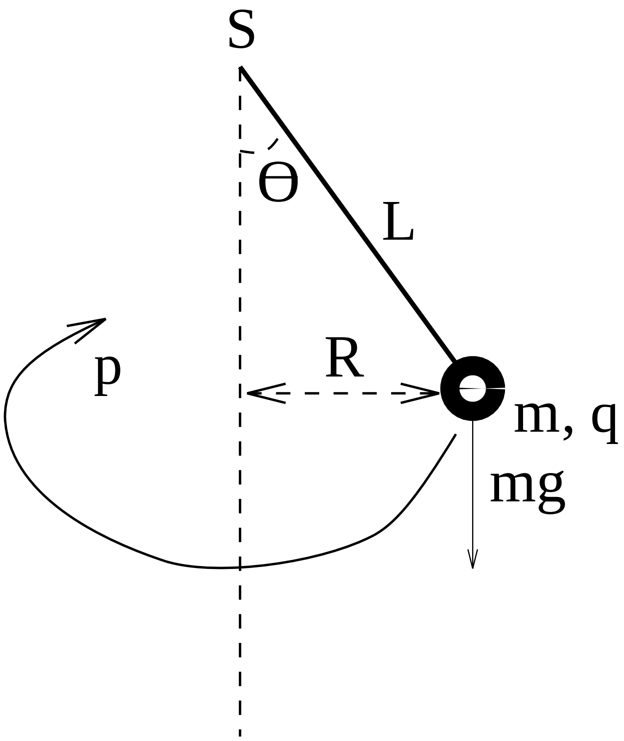
1 (12 pts) Quantization, Uncertainty Principle, Classical Correspondence: Conical Pendulum

In this problem, you will analyze the consequences of the de Broglie relations (i.e., Bohr quantization) on the motion of a "conical pendulum." A conical pendulum (see Figure 1) is very much like a normal pendulum, there is a mass m suspended from a point S by a rigid massless rod of length L and subject to the force of gravity mg. In the case of the conical pendulum, the rod is free to move in anyway about the point S so that it can make circular orbits of radius R as indicated in the figure. For the purposes of this problem we will also place, a small test charge q on the mass m. For reference, we will call the momentum of the particle p, the angle which the rod L makes with the direction of the vertical Θ, the distance from the mass m to the vertical line R, the total energy (kinetic plus potential) of the pendulum E, and the period when the particle undergoes circular motion T. In the Bohr analysis of the motion there must be an integral number n of de Broglie wavelengths around the circumference of the orbit of radius R. In a quantum state n, there are specific values allowed to the angle and energy of the system, which we shall call Θ_n, and E_n, respectively. Your task is to use the Bohr quantization condition to compute Θ_n and E_n, compare the lowest state $n = 1$ with the predictions of the Heisenberg Uncertainty Principle, and understand the behavior of the level spacings in the spectrum E_n by using the correspondence principle.

a) Intuitively, how to you expect the period of the conical pendulum's motion to vary (increase or decrease) as its energy it increased? Using the general result from the correspondence principle given in class that in the classical limit, the level spacing is given by $\Delta E \longrightarrow h/T_{classical}$, where $T_{classical}$ is the classical period of the motion, how do you expect the spacing between the energy levels of the pendulum to vary (increase or decrease) as its energy increases?

b) What is the momentum p of the particle as a function of the angle Θ of the pendulum as the particle moves in its circular orbit? What is the period and total energy (kinetic plus potential) of this motion? Express all three answers in terms of just Θ, L, m and g.

c) If a small test charge q is placed on the mass m, what frequency radiation would the pendulum emit when rotating in a circular orbits at an angle Θ?



d) Use the Bohr quantization condition that there must be an integral number n of de Broglie wavelengths around the circumference of the orbit of radius R, to find a condition on the allowed quantized values of Θ, Θ_n in terms of the quantum number n and the constants $\hbar$, m, g and L. (Hint: To get an explicit form for Θ_n you would have to solve a ugly equation. Instead you may express your result in the implicit form $\sqrt{\sin^3\Theta\tan\Theta} = f(\alpha, n)$ where $\alpha^2 \equiv \hbar^2/(m^2gL^3)$ and f is some function.

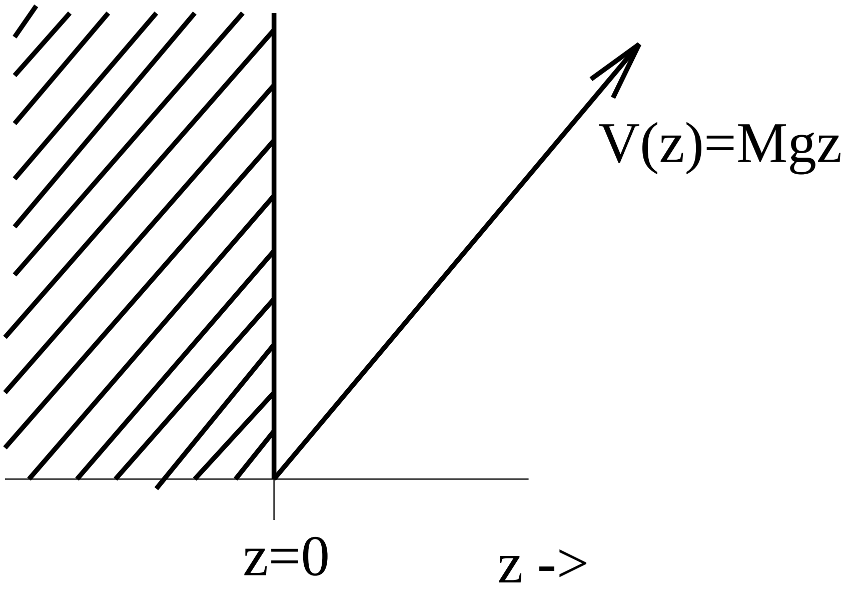
e) Check your result for **d)** by using the *uncertainty principle* to to estimate the average angle Θ_1 in the ground state of the pendulum. (For this part of the problem you may assume that Θ is small so that $\sin\Theta \simeq \Theta$.)

f) Under the assumption that $\hbar << \sqrt{m^2L^3g} \Rightarrow \alpha << 1$ (the classical limit), give an expression for the spacing $\Delta\Theta = \Theta_n - \Theta_{n-1}$ between the allowed quantized angles of the system in terms of α and the classical angle of the pendulum Θ. From your expression for $\Delta\Theta$, determine the spacing $\Delta E \equiv E_n - E_{n-1}$ between neighboring energy levels in terms of the angle of the pendulum Θ. Do the spacings exhibit the behavior you predicted in a)?

g) Show explicitly that photons emitted by the pendulum making a transitions between neighboring energy levels $n \longrightarrow n-1$ represent precisely the same frequency radiation expected in **c)**.

2 (10 pts) Quantization, Uncertainty Principle, Classical Correspondence: Bouncing Ball

In this problem, you will analyze the motion of a bouncing ball of mass m subject to the gravitational force mg. You are now able to treat this problem at a level far beyond that you used in Quiz I. You will now investigate the consequences of the de Broglie relations (i.e., Bohr-Sommerfeld quantization) on the motion of the ball. A sketch of the potential experienced by the ball is given in Figure 2.



The bouncing of a ball is a periodic motion; we shall call the period of the motion T. The Bohr-Sommerfeld quantization condition

$$\oint p\,dq = nh, \text{where } n = 1, 2, ...,$$

restricts the allowed energies of the bouncing ball to a discrete set of energies E_n indexed by the quantum number n. Your task again is to compute the spectrum of allowed energies E_n, and to verify that the ground state gives results consistent with the expectations of the uncertainty principle and that the large n limit is consistent with the expectations of classical physics.

a) Without performing an analysis, how to you expect the period of the ball's bouncing to vary (increase or decrease) as the energy of its motion increases? What does this then tell you about the spacing of the energy levels for large n?

b) Compute the period of the ball's motion T in terms of its total energy E.

c) If a small test charge q is placed on the mass m, what frequency radiation would this bouncing ball emit?

d) Use the Bohr-Sommerfeld quantization condition to find the allowed energy levels of the ball. Sketch the energy spectrum on an energy-level diagram. Show that the spacings for large n behave as you expected in **a)**.

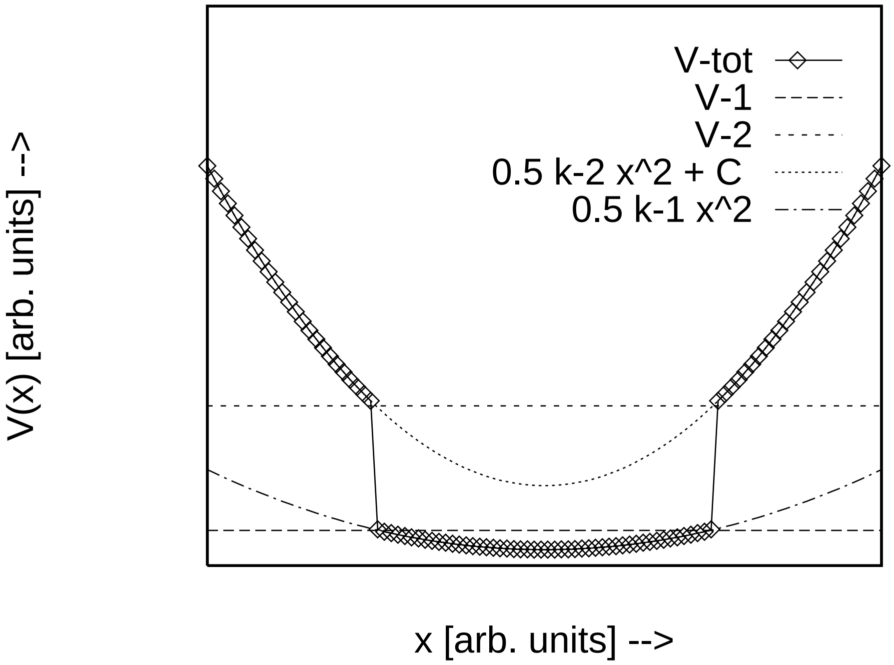
e) Check your result for **d)** for the case $n = 1$ by using the *uncertainty principle* to to estimate the ground state energy of the ball, E_1.

f) Show explicity that, in the classical limit, the period of the motion of the ball T is exactly consistent with the energy-level spacings.

3 (7 pts) Power of the Correspondence Principle

Note that no computations are required for this problem.

Use the correspondence principle to sketch the energy spectrum of the allowed states E_n of a particle of mass m moving in the potential V_{tot} indicated in the figure below. The potential (marked by diamonds in Figure 3) consists of two simple harmonic oscillator potentials of spring constants



k_1 and k_2 ($k1 < k_2$) pasted together with a discontinuity bridging the energies V_1 and V_2 indicated in the figure. The level of $V = 0$ corresponds to the minimum potential in the plot.

Be sure to include the regions $E < V_1$, $V_1 < E < V_2$ and $V_2 < E$ in your sketch. For the purposed of the sketch you may assume that there are five or so levels in each of the ranges $E < V_1$ and $V_1 < E < V_2$. Be sure not to forget the zero point energy of the first level.

4 (15 pts) Fourier Transforms

As we have discussed in lecture, there is a close relationship between the measurement of position and momentum and the mathematical concept of the *Fourier Transform*. Many of the known mathematical properties of the Fourier Transform correspond directly to physical concepts which we have only yet been able to discuss *qualitatively*. To be able to discuss these physical concepts on a quantitative level, we must develop a working knowledge of the Fourier transform. The purpose of this problem is to develop just such knowledge.

The operation of the Fourier transform we shall write as

$$\begin{aligned} \psi(x) &= \mathcal{F}\{\hat{\psi}(k)\} \\ &\equiv \int \frac{dk}{\sqrt{2\pi}}\, \hat{\psi}(k)\, e^{ikx}. \end{aligned} \tag{1}$$

The inverse relationship is

$$\begin{aligned} \hat{\psi}(k) &= \mathcal{F}^{-1}\{\psi(x)\} \\ &\equiv \int \frac{dx}{\sqrt{2\pi}}\, \psi(x)\, e^{-ikx}. \end{aligned} \tag{2}$$

For this problem you will consider a wave packet where

$$\hat{\psi}(k) = N e^{-ikx_0} e^{-D|k-k_0|}. \tag{3}$$

a) Compute the value N such that $\hat{\psi}(k)$ is properly normalized and use this value for the rest of the problem.

b) Compute $\psi(x)$ from $\hat{\psi}(k)$ using the definition (1) and verify that $\psi(x)$ is properly normalized so long as $\psi(k)$ is properly normalized.

c) Compute $\hat{\psi}(k)$ from your expression for $\psi(x)$ using the definition (2) and verify that you recover (3).

d) Sketch the real part of both $\psi(x)$ and $\hat{\psi}(k)$, indicating in your sketches, in terms of the parameters x_0, k_0 and D, the approximate width and center of each function and the wavelength of any oscillations.

e) Confirm that your results satisfy the general Fourier theorem on the *scaling* of functions: if $f(x) = \mathcal{F}\{\hat{f}(k)\}$, then $\mathcal{F}\{\hat{f}(sk)\} = \frac{1}{s} f(x/s)$. State what this means in terms of the behavior of the functions $\hat{f}(sk)$ and $f(x/s)$ as the parameter s varies. Interpret your result physically in terms of the uncertainty principle.

f) Confirm that your results satisfy the general Fourier theorem on the *displacement* of functions: if $f(x) = \mathcal{F}\{\hat{f}(k)\}$, then $\mathcal{F}\{\hat{f}(k-k_0)\} = e^{ik_0x} f(x/s)$. Interpret this result physically in terms of the effect of this operation on the momentum distribution and on the oscillatory behavior of the packet in real space.

g) Confirm that your results satisfy the general Fourier theorem on the *multiplication by a phase* of functions: if $f(x) = \mathcal{F}\{\hat{f}(k)\}$, then $\mathcal{F}\{\hat{f}(k)e^{-ikx_0}\} = f(x-x_0)$. How is a shift in the position of the center of the wave packet reflected in the momentum wavefunction $\hat{\psi}(k)$? How is it reflected in the momentum probability distribution $|\hat{\psi}(k)|^2$?

h) Confirm, for the case $n=1$, that your results satisfy the general Fourier theorem on the *moment-derivative relationship*: if $f(x) = \mathcal{F}\{\hat{f}(k)\}$, then $\mathcal{F}\{(ik)^n \hat{f}(k)\} = \frac{d^n}{dx^n} f(x)$. Give an interpretation of this result in terms of the *change* which $f(x)$ undergoes under an infinitesimal displacement δ, $\Delta f(x) \equiv f(x) - f(x-\delta) \longrightarrow f'(x)\delta$ *and* the corresponding operations on $\hat{f}(k)$.

i) Compute $< X >$, $< P >$ $(=< \hbar k >)$, ΔX and ΔP for this wave packet. Confirm that the uncertainty principle $\Delta X \, \Delta P \geq \hbar/2$ holds in this case.

You may find the following integrals useful:

$$\int_{-\infty}^{\infty} ds \, \frac{cos(ps)}{1+s^2} = \pi e^{-|p|}$$

$$\int_0^{\infty} ds \, s^n \, e^{-s} = n!$$