Massachusetts Institute of Technology

Department of Physics

Physics 8.04 October 12, 1995

Quantum Physics I, Spring 1995

Ajenda and **readings** for the week of April 3:

Tuesday April 4, Lecture 13 —
General Features of the Time Dependent Scrodinger Equation (TDSE) & Separation of Variables, Time Independent Schrodinger Equation (TISE), Completeness and Orthogonality of Eigenstates: **Lecture Notes**.

Thurs. April 6, Lecture 14 —
Variational Principle, General Virial Theorem, Qualitative solultions of TISE: **Lecture Notes, FT 3.11**.

Problem Set VI

(Due Friday, April 7 at 5:00 pm sharp.)

Work requested of the students is highlighted by either a question mark or underlining.

1 (8 pts) Gaussian Wavepacket I: The Transform

In the next problem you will be asked to consider, in the *position* representation, the physics of a wave packet given in the momentum representation by

$$\tilde{\psi}(k,t) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \exp\left(-\left(\frac{\alpha}{2}(k-k_0)^2 + \frac{i\hbar}{2m}k^2 t\right)\right)$$

$$\psi(x,t) = \int \frac{dk}{\sqrt{2\pi}}\,\tilde{\psi}(k,t)e^{ikx}.$$

(The notation $\exp(x)$ just means e^x.) In this problem, you first find an explicit form for the wavefunction in the position representation.

1.1 a)

By completing the square in the Gaussian integral as in the lecture notes, <u>show</u> that

$$\psi(x,t) = \frac{\left(\frac{1}{\pi\alpha}\right)^{\frac{1}{4}}}{\sqrt{\left(1+\frac{i\hbar t}{m\alpha}\right)}} \exp\left\{\frac{-\frac{x^2}{2\alpha} + ik_0 x - \frac{i\hbar k_0^2 t}{2m}}{1+\frac{i\hbar t}{m\alpha}}\right\}.$$

1.2 b)

Show also that

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{\pi\alpha\left(1+\left(\frac{\hbar t}{m\alpha}\right)^2\right)}} \exp\left\{-\frac{(x-\frac{\hbar k_0 t}{m})^2}{\alpha(1+\left(\frac{\hbar t}{m\alpha}\right)^2)}\right\}$$

Hints:
Note that you will need to use the result

$$I(y) \equiv \int e^{-\alpha(x+iy)^2}\,dx = \sqrt{\pi/\alpha}.$$

If you have studied complex variables, you obtain this by shifting the contour. If not, you can take $\frac{d}{dy}I(y)$ and by noting that the result may be put in the form of an integral of an exact derivative, prove $\frac{d}{dy}I(y) = 0$ and hence $I(y) = \text{ const } = I(0)$. Finally, you show that $I(y=0) = \sqrt{\pi/\alpha}$ using the standard trick of changing to polar coordinates to show $(I(y=0))^2 = \left(\int dx\, e^{-\alpha x^2}\right)\left(\int dy\, e^{-\alpha y^2}\right) = \int 2\pi r\, dr\; e^{-\alpha r^2} = \pi/\alpha$.

Also, you will face manipulations of complex numbers such as

$$\left|\frac{1}{a+bi}\right|^2 = \frac{1}{a+bi}\cdot\frac{1}{a-bi} = \frac{1}{a^2+b^2}$$

and

$$\exp\left\{\frac{1}{a+bi}\right\} = \exp\left\{\frac{1}{a+bi}\right\}\exp\left\{\frac{1}{a-bi}\right\} = \exp\left\{\frac{1}{a+bi}+\frac{1}{a-bi}\right\} = \exp\left\{\frac{2a}{a^2+b^2}\right\}$$

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2 (12 pts) Gaussian Wavepacket II: The Physics

2.1 a)

Show explicitly that the wavepacket in problem I is a solution to the time dependent Schrodinger equation for a particle moving in free space $V(x) = 0$,

$$i\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2\psi(x,t).$$

2.2 b)

Determine $< x >$ and $< p >$ as a function of time and explain how your results are consistent with Ehrenfest's theorem,

$$\frac{d}{dt} < x > = < \frac{p}{m} >$$
$$\frac{d}{dt} < p > = < -\frac{d}{dx}V > .$$

(Hint: You should not have to integrate, the values of $< x >$ and $< k >$ should be clear from the forms for $|\psi(x,t)|^2$ and $|\tilde{\psi}(k,t)|^2$.

2.3 c)

Compute Δx and Δp as a function of time t. Give a *physical* interpretation for the behaviors you find for Δx and Δp and describe how the packet maintains consistency with the Heisenberg uncertainty principle as it evolves in time.

2.4 d)

Compute the probability current

$$j(x,t) \equiv \frac{\hbar}{2im}\left(\psi(x,t)^* \partial_x \psi(x,t) - \psi(x,t)\partial_x \psi(x,t)^*\right)$$

for the Gaussian wavepacket. (Hint: to reduce your writing, you may wish to give special names to combinations of variables that do not depend on time.)

2.5 e)

Show that the probability $\mathcal{P}(x,t) \equiv |\psi(x,t)|^2$ is conserved by verifying directly

$$\partial_t P(x,t) + \partial_x j(x,t) = 0.$$

3 (12 pts) Coherent State of Simple Harmonic Oscillator

To investigate Ehrenfest's Theorem in a non-trivial case, consider a particle of mass m in a one-dimension simple Harmonic oscillator of frequency ω. The Hamiltonian for this system is then given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2.$$

In this problem we will deal with a special class of solutions to the time dependent Schrodinger for the harmonic oscillator known as "coherent" states. These solutions all have the form

$$\psi(x,t) = \exp\left(-A(t)x^2 + B(t)x + C(t)\right). \tag{1}$$

3.1 a)

Write down the time dependent Schrodinger equation which $\psi(x,t)$ obeys in this problem. Show that there are solutions of the form (1) provided that A, B and C obey the differential equations,

$$A'(t) = \frac{i}{2\hbar m}\left(m^2\omega^2 - 4\hbar^2 A(t)^2\right)$$

$$B'(t) = -\frac{2\hbar i}{m}A(t)B(t)$$

$$C'(t) = \frac{i\hbar}{2m}\left(B(t)^2 - 2A(t)\right).$$

3.2 b)

Find the value $A(t) = A_0$ for which the solution for $A(t)$ is a constant and does not vary with time. Find general solutions for $B(t)$ and $C(t)$ when $A(t)$ takes this value.

3.3 c)

Compute $\mathcal{P}(x,t) = |\psi(x,t)|^2$ from your solution and show that the probability distribution does not change shape but simply shifts rigidly back and forth as a function of time. Stated mathematically, this means that we may write $\mathcal{P}$ in the form $\mathcal{P}(x,t) = \mathcal{P}\left(x - x_0\cos(\omega t + \phi_0)\right)$, where x_0 and ϕ_0 are constants independent of time. The fact that the distribution keeps its shape as a function of time is related to why these solutions are called coherent states.

3.4 d)

Verify that your results are consistent with Ehrenfest's theorem,

$$\frac{d}{dt} < x >=< \frac{p}{m} >$$

$$\frac{d}{dt} < p >=< -\frac{d}{dx} V >$$

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3.5 e)

By reasoning from Ehrenfest's theorem alone, <u>show</u> that for the case of the simple harmonic oscillator, regardless of the initial value of the wavefunction, the average expectation of position and momentum *always* undergo simple harmonic motion:

$$< x >= x_0 \cos(\omega t + \phi_0),$$

$$< p >= -m\omega x_0 \sin(\omega t + \phi_0),$$

for some fixed amplitude x_0 and phase ϕ_0.