Massachusetts Institute of Technology

Department of Physics

Physics 8.04 October 12, 1995

Quantum Physics I, Spring 1995

Agenda and **readings** for the week of April 3:

Tuesday April 11, Lecture 13 —
Qualitative solutions cont'd (still, **FT 3.11**).

*Quantit*ative Solutions:

- Infinite Square Well: **FT 3.4-3.5**
- Finite Square Well: **FT 4.2**
- delta-function: **Lecture Notes**

Thurs. April 13, Lecture 14 —
Transfer Matrix Approach **Lecture Notes**
Simple Harmonic Oscillator **FT 4.3**

Problem Set VI

(Due Friday, April 14 at 5:00 pm sharp.)

Work requested of the students is highlighted by either a question mark or underlining.

1 (10 pts) Eigenstate "Expansion"/Why operators are nice

In this problem you will need to use the functional form for the eigenstates of a particle of mass m in a rigid "box" (infinite square-well potential) of width W. Properly normalized, these wavefunctions are

$$\psi_n^{(W)}(x) = \sqrt{\frac{2}{W}} \sin\left(\frac{n\pi}{W}x\right); \quad n = 1, 2, ...,$$

with energies

$$E_n^{(W)} = \frac{\hbar^2\pi^2}{mW^2}n^2; \quad n = 1, 2, ...$$

Now imagine a particle of mass m in the ground state, $\psi_1^{(L/2)}$, in an infinite well of width $L/2$. (You might imagine preparing this state by putting the particle in the well and then waiting a very long time for it to radiate away all of its excess energy.) Suddenly, the right wall of the well is moved from $x = L/2$ to $x = L$, and the energy of the particle is measured immediately, before the quantum state of the particle has an opportunity to respond to the sudden change in the environment.

1.1 a)

Write down the wavefunction $\psi_{exp}(x)$ of the particle immediately after the expansion.
(Hint: What is the the probability of finding the particle at a position $x > L/2$ *immediately* after the expansion of the well?)

1.2 b)

Apply the superposition principle to write the wave function of the particle immediately after the expansion in terms of the eigenstates of the new, widened well as

$$\psi_{exp}(x) = \sum_n c_n \psi_n^{(L)}(x).$$

Determine the constants c_m in the sum. (Hint: exploit the orthogonality condition $(\psi_m, \psi_n) = \delta_{mn}$ to determine $c_m = -4\sqrt{2}(\sin\frac{m\pi}{2})/\pi/(m^2-4)$, and note $\lim_{m\to 2} c_m = \sqrt{2}/2$.)

1.3 c)

What is the probability that, after the expansion, the particle will be found to have an energy $E_1^{(L)}$, equal to the ground state energy of the widened well? What is the probability that the particle will be found with energy $E_2^{(L)}$? With energy $(E_1^{(L)} + E_2^{(L)})/2$?

1.4 d)

For all values of $n = 1, 2, ...$, make a plot of the probability P_n of finding the particle with each of the energies $E_n^{(L)}$.

1.5 e)

Compute the average energy directly from the sum

$$< E > \equiv \sum_{n=1}^{\infty} P_n E_n^{(L)}.$$

How does this result compare to the energy of the particle before the well was widened? Give a physical explanation for this comparison.

Hint: you will need the infinite sum

$$\sum_{n=1,3,5,\ldots} \frac{n^2}{(n^2-4)^2} = \frac{\pi^2}{16},$$

which may be obtained from the hypergeometric function.

1.6 f)

Verify that the same result for $< E >$ is obtained easily (without having to perform infinite summations or know about hypergeometric functions) using the operator expression

$$< E > = \int dx\, \psi_{exp}^*(x) \hat{H} \psi_{exp}(x) = \int dx\, \psi_{exp}^*(x) \left(-\frac{\hbar^2}{2m}\partial_x^2\right) \psi_{exp}(x).$$

2 (10 pts) Time Dependent Solutions in the Square Well

At time $t = 0$ a particle of mass m in an infinite square well of width L is in a state described by

$$\psi(x) = \frac{1}{\sqrt{L}}\left(\sin\frac{\pi}{L}x + \sin\frac{2\pi}{L}x\right).$$

2.1 a)

Determine the wavefunction of this particle at times $t > 0$ and sketch a plot of the probability distribution $\mathcal{P}(x,t)$ for $t = 0$ and $t = t_0 = \frac{2}{3}\frac{mL^2}{\pi\hbar}$.

2.2 b)

How much probability has "flowed" from the left half of the well, $x < L/2$, to the right half of the well, $x > L/2$, during the interval $0 < t < t_0$?

2.3 c)

Use the probability current to compute as a function of time t the rate $I(t) \equiv \frac{d\mathcal{P}}{dt}$ at which probability flows across the point $x = L/2$ from the left half to the right half of the well. Integrate this result from $t = 0$ to $t = t_0$ and compare with your result in **b)**.

3 (10 pts) Proofs using the Eigenstate Expansion

As discussed in lecture, the time dependence of *any* quantum state may be written as

$$\Psi(x,t) = \sum_{n=0}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

$$\Phi(x,t) = \sum_{n=0}^{\infty} d_n \phi_n(x) e^{-iE_n t/\hbar},$$

in terms of normalized stationary states ψ_n of energy E_n. ($\hat{H}\psi_n = E_n\psi_n$, $(\psi_n, \psi_m) = \delta_{nm}$.) This fact is extremely useful and may be used to prove many theorems easily. Use it to prove the following:

(Hint: the proofs follow by direct substitution of the above expansions for Ψ and Φ.)

3.1 a)

Parseval's Theorem for the eigenstate expansion:

$$\sum_n d_n^* c_n = (\Phi(x, t=0), \Psi(x, t=0)).$$

Also show that $\Psi(x, t=0)$ and $\Phi(x, t=0)$ are properly normalized if and only if $\sum_n |c_n|^2 = 1$ and $\sum_n |d_n|^2 = 1$, respectively. Throughout the rest of this problem, assume that these normalization conditions hold.

3.2 c)

The variational principle:
The average energy (of a properly normalized state) always exceeds the ground state energy

$$< E > \equiv (\Psi(x, t = 0), \hat{H}\Psi(x, t = 0)) \geq E_0.$$

From this result, show also that for any function $\Theta(x)$ (even if not properly normalized), that the Rayleigh quotient obeys

$$\mathcal{R}[\Theta] = \frac{(\Theta, \hat{H}\Theta)}{(\Theta, \Theta)} \geq E_0.$$

3.3 d)

$\Psi(x, t)$ of the form above satisfies the time dependent Schrödinger equation (TDSE):

$$i\hbar\partial_t\Psi(x, t) = \hat{H}\Psi(x, t).$$

3.4 e)

The conservation of probability:
$\mathcal{P}(x, t) \equiv |\Psi(x, t)|^2$ is conserved:

$$\int dx\, \mathcal{P}(x, t) = 1 \quad \text{for all times } t > 0.$$

3.5 f)

The conservation of energy:
The expectation of the energy energy associated with a wavefunction $\Psi(x, t)$ evolving according to the TDSE obeys:

$$< E >_t \equiv (\Psi(x, t), \hat{H}\Psi(x, t)) = \text{constant}.$$

4 (8 pts) Qualitative Plots of a Wavefunction

Do Problem 3.13 from *French and Taylor.*

5 (8 pts) Practice in Constructing Qualitative Plots

Do Problem 3.15 (a)(b)(d)(e) from *French and Taylor.*