Massachusetts Institute of Technology

Department of Physics

Physics 8.04 May 18, 1996

Quantum Physics I, Spring 1996

Agenda for the week of May 7:

- L21 05/07: Resonant scattering from a square done with Feynman diagrams.
- L22 05/09: Double barrier well (done with Feynman renormalization/dressed interactions) and discussion of metastable states.

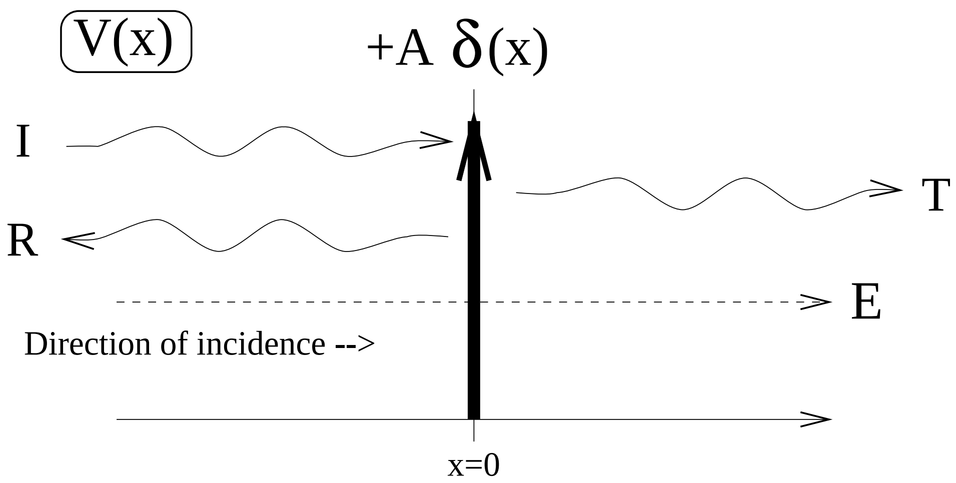
Problem Set 10 – Last one! :-(

(Due Friday, May 10 at 5:00 pm sharp.)

Work requested of the students is highlighted by either a question mark (?) or underlining.

1 Delta-function Scattering/Decay of Metastable states

Consider the scattering generated by a repulsive Dirac δ-potential of strength A as in the figure below.



A steady stream of N particles per second is incident on the δ-potential from the left. Each particle has mass m and energy E and is incident as shown.

1.1 Reflection and transmission amplitudes and probabilities from a delta-potential

Derive the transmission and reflection quantum amplitudes $t(k)/s(k)$ and $r(k)/s(k)$, respectively, and from them determine the transmission and reflection probabilities, P_t and P_r respectively. Give your results in terms of the parameter $\eta \equiv \frac{2mA}{\hbar^2 k}$. Sketch your results for P_r and P_t as function of η.

Interpret, in physical terms, your findings in the limits $\eta \to -\infty$, $\eta \to 0$, $\eta \to +\infty$.

1.2 Time lags

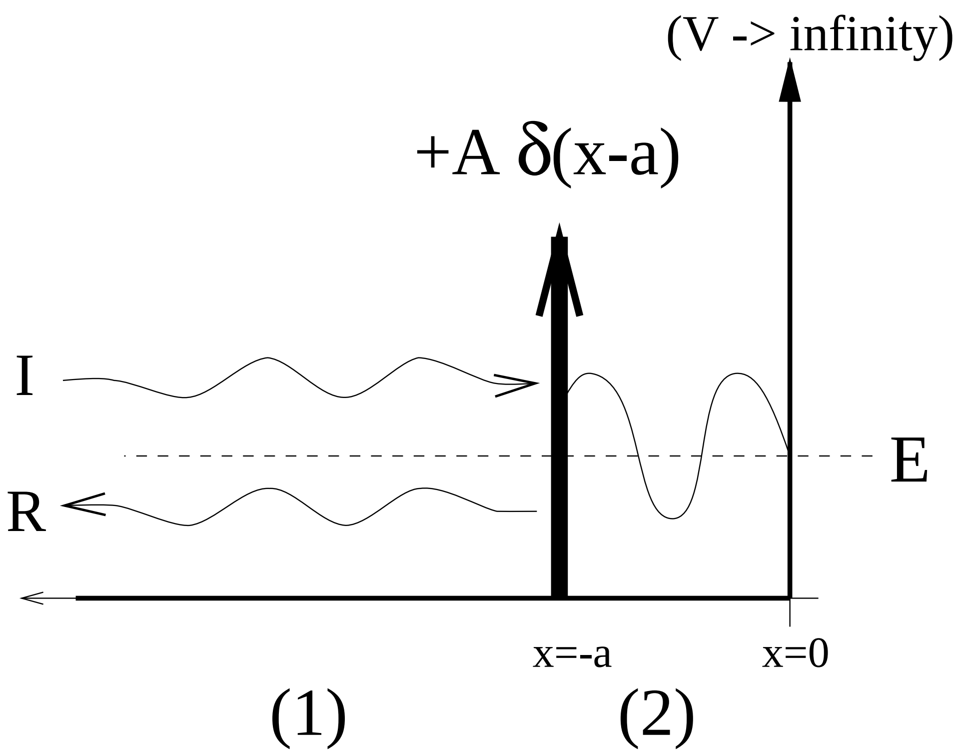
Now, a left-incident *wave packet* is sent toward the δ-function. The packet is initially located at position $x_o < 0$ at time $t = 0$.

Compute the time required for the reflected packet to return to position x_o. How does this time compare with $2x_o/v_{cl}$, in other words, what is the time lag, if any, in the reflected packet?

Compute the time required for the transmitted packet to reach position $-x_o > 0$. How does this time compare with $2x_o/v_{cl}$, in other words, what is the time lag, if any, in the transmitted packet?

2 Metastable stable solutions to the TISE

Now consider the problem below, where the δ-function is centered at $x = a$ instead of $x = 0$ and the potential is infinite for $x > 0$.



2.1 Solution of the TISE

Find the solutions of the TISE for this system consistent with left-incident boundary conditions.

2.2 Build-up of trapped particles

Sketch for large values of A, the ratio, as a function of k, of a) the probability of finding the particle trapped between the δ-function and the infinite wall to b) to the rate of particles incoming from the

source. For a fixed value of the incoming beam, near which values of k do you find the greatest build up of particles?
Note: to make the algebra simpler, you may work under the limit that $ka >> 1$ so that $\int_{-a}^{0} \sin^2 kx\, dx \approx a/2$.

2.3 Solutions to the TDSE: Time Delay for Metastable scattering

Examining the phase shift between the incoming and out going waves, use the method of stationary phase to compute the time lag of the outgoing packet as a function of k. Sketch, for large A, the time lag as a function of k. For which values of k do you find the maximum value of the time lag?

3 Feynman diagrams: scattering from a double delta well

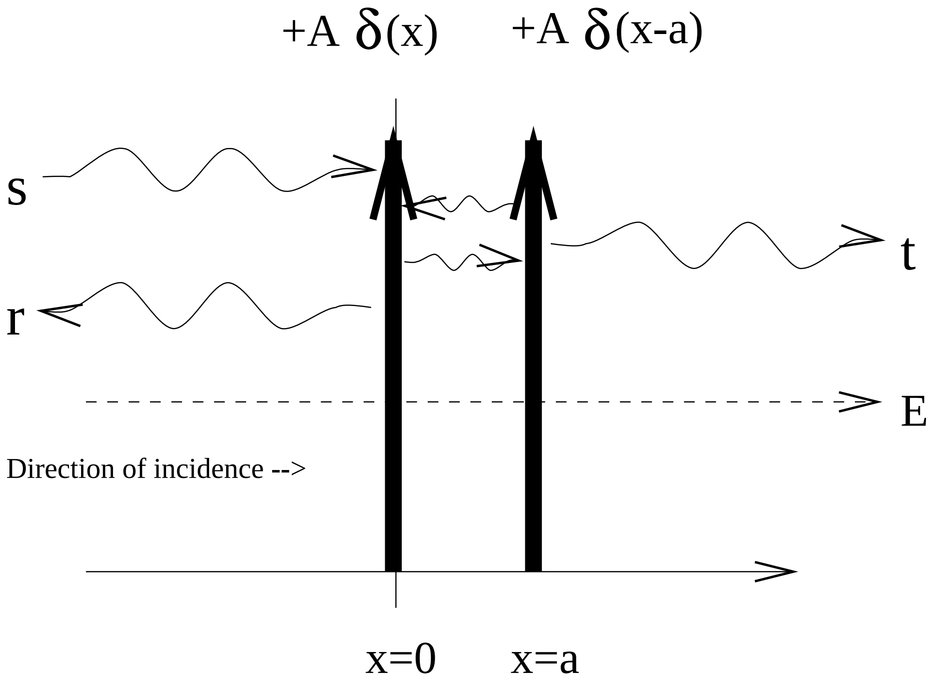
3.1 Derive your own Feynman Rules

State the correct Feynman rules for diagrams describing propagation between and scattering from δ-functions of strength $+A$, giving expressions for all of the factors that appear in such diagrams. Express your results terms of 1) k, 2) d, the distance traveled between δ functions, and 3) the parameter η defined in (1).

Hint: You should be able to find these easily from your results in (2).

3.2 Double delta potential

Use your Feynman rules to compute the transmission and reflection amplitudes $t(k)/s(k)$ and $r(k)/s(k)$ for a pair of δ-functions, both of strength $+A$, at a separation of a as in the figure below.



Plot the transmission amplitude as a function of k, indicating the points corresponding to resonant tunneling.

4 First Look at Angular Momentum

4.1 The operator for Lz

Background: Associated with a particle moving in the x-y plane is the observable of its angular momentum in the z-direction,

$$L_z = xp_y - yp_x.$$

Write down the associated quantum operator $\hat{L}_z$ in the x-y representation.

Using the chain rule, show that under the change of variables

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi, \end{aligned}$$

your $\hat{L}_z$ operator may be rewritten as

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}.$$

4.2 Pure states of Lz

Consider now a particle which is constrained to move on a circle of fixed radius R in the x-y plane, so that its position is entirely determined by the angle ϕ. From the form of your L_z operator and using the quantum purity test, show that the wavefunction of the particle in a pure state of angular momentum with value L_z is

$$\psi(\phi) = e^{iL_z\phi/\hbar}. \tag{1}$$

Because ϕ and $\phi + 2\pi$ represent the same point, we expect that $\psi(\phi + 2\pi) = \psi(\phi)$. Realizing this, what then are the allowed values of L_z in your pure states in 1?

4.3 Pure states of Energy for a particle on a ring

The Hamiltonian for a particle of mass m moving along a ring of radius R is

$$\hat{H} = \frac{\hat{L}^2}{2I},$$

where I is the moment of inertia of the particle moving on the circle, $I = mR^2$. Using the the same boundary conditions as before, $\psi(\phi + 2\pi) = \psi(\phi)$, determine the allowed energies for the rotating particle.