Massachusetts Institute of Technology

Department of Physics

Physics 8.04 February 26, 1996

Quantum Physics I, Spring 1996

Agenda and **readings** for the week of Feb. 20:

(Note: The abbreviations for the readings are RH = *Resnick and Halliday*, K = *Kroemer*, FT = *French and Taylor*, CN = class notes given out after lecture (also appears on web after lecture).)

- L6, 02/27: Bohr Model; Atomic Spectra, Franck Hertz, Semiclassical quantization, Moseley X-ray Spectra
  Readings: RH 4.7,7.3,7.5,7.7; CN "Bohr-Sommerfeld Quantization and the Classical Limit"; FT 1.10
- L7 02/29: Formal Theory; Quantum State, Pure states, superposition, wavefunctions, position and momentum representations. **Readings: CN "Quantum States and Observables"**

Problem Set 3

(Due Friday, March 1 at 5:00 pm sharp.)

Work requested of the students is highlighted by either a question mark (?) or underlining.

1 HUP: Limitations on Predictability of Trajectories

1.1 Spreading of a wave packet

Rederive the result shown in class, that the minimum uncertainty $\Delta x(t)$ in the position of a particle at time t grows as

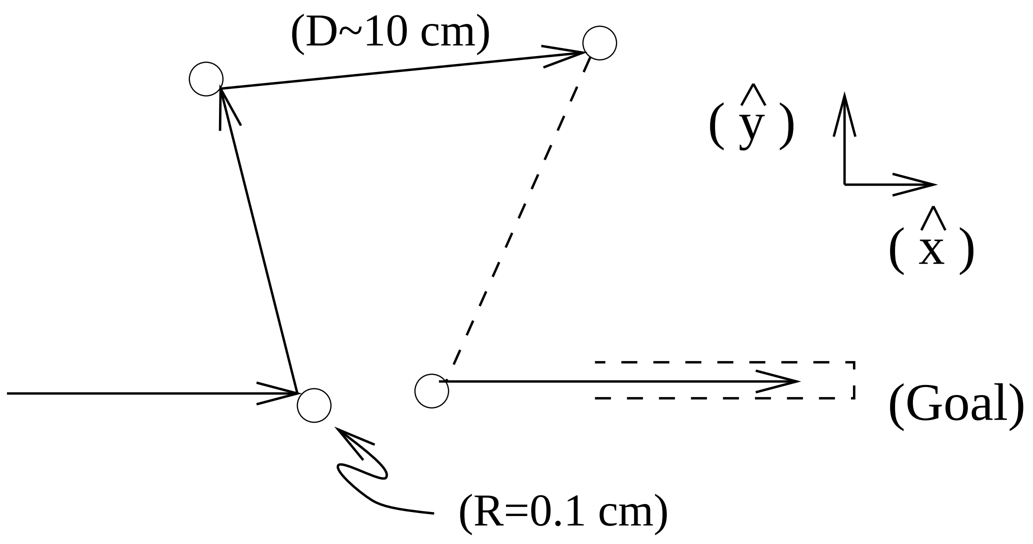
$$\Delta x(t) = \sqrt{\Delta x(0)^2 + \left(\frac{\hbar t}{2m\Delta x(0)}\right)^2}.$$

Verify explicitly that this result holds exactly in the case of the wave packet in problem 2.4 (problem 4 of PS # 2) if we define Δx as the distance from the center of the packet $(x_0 + \hbar k t/m)$ to the point where the envelop decays to $e^{-1/4}$ of its original value.

1.2 Quantum Pin Ball

In this problem you will investigate the feasibility of quantum uncertainty effects playing a role in a simplified version of Pin Ball.

In this version of Pin Ball, the table is flat so that gravity plays no role, and there is a series of small pins (of radius $R = 0.1cm$) at an average spacing of $D = 10cm$. The object is to hit a small ball (mass $m = 10g$, radius small enough to ignore) at a horizontal velocity ($v = 10cm/s$) horizontally (along $\hat{x}$) at just the right angle and height y so that it collides with a predetermined series of pins and eventually enters the goal. (See figure below.) You will show that after a relatively small number of pins, even under idea circumstances, it is impossible to do this on a regular basis. Such regular behavior is forbidden by the Heisenberg Uncertainty Principle (HUP) of quantum mechanics.



1.3 Uncertainty in ball's direction after the first collision

Establish the limitation which the uncertainty principle places on the precision with which the angle leaving the first pin may be set. Assuming that the uncertainties in the first collision are small, show that the uncertainty in the direction leaving the first pin is on the order of

$$\Delta\theta_1 \approx \sqrt{(\Delta y/R)^2 + (D\Delta p_y/Rp)^2},$$

where $p = mv$ is the magnitude of the momentum the Pin Ball. From this and the HUP, determine the least possible uncertainty in the angle θ_1 leaving pin #1.

(Note that the figure shows an exaggerated case where the uncertainties $\Delta\theta_0$ and Δy result is a rather large uncertainty $\Delta\theta_1$.)

1.4 Magnification of errors

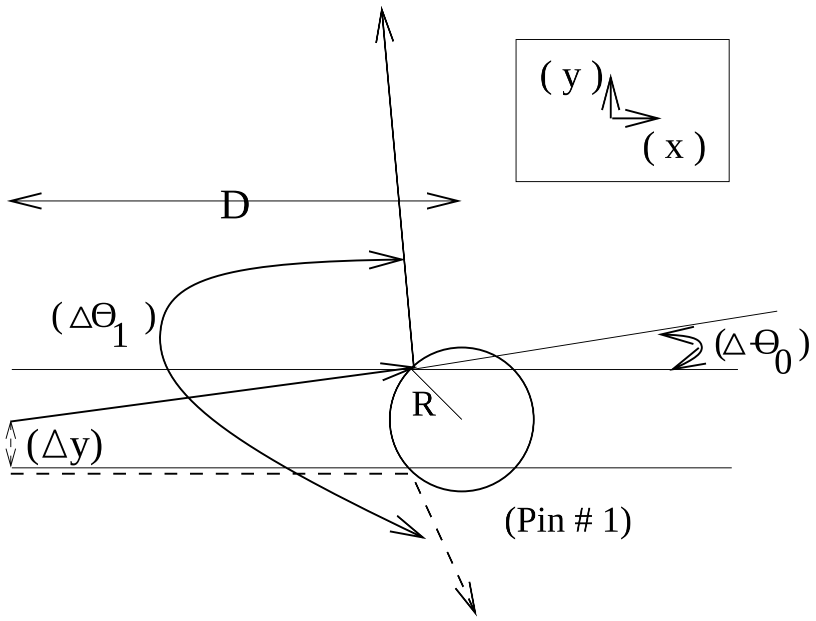
Show, again in the limit of small uncertainties, that an uncertainty in the ball's direction of $\Delta\theta_n$ at the n^{th} pin leads to an uncertainty in its direction upon leaving the next pin of $\Delta\theta_{n+1} \approx (D/R)\Delta\theta_n$. (To obtain this result, you may assume $R << D$).

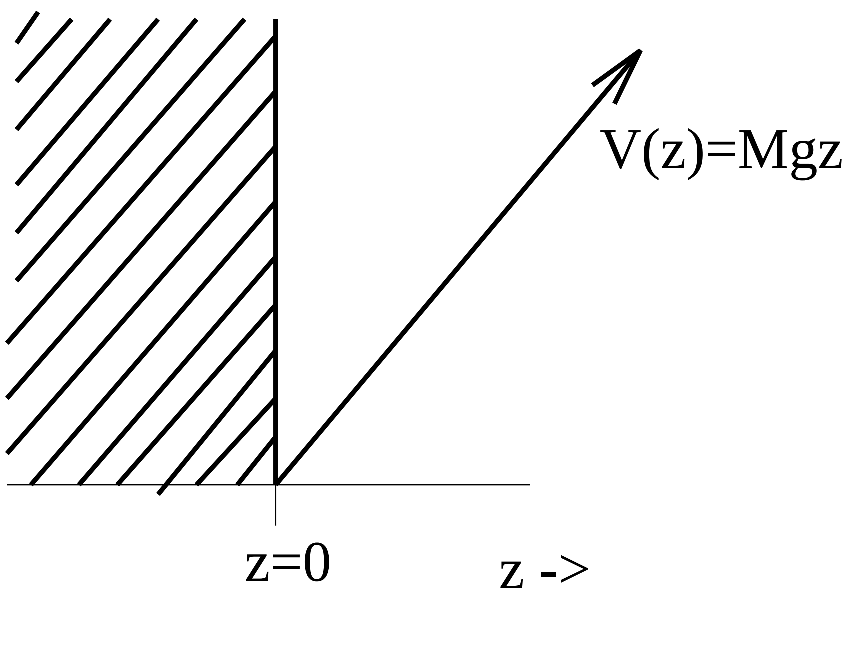
1.5 Perfect gambling device

Given your results in 1.3 and 1.4, how many collisions would the ball have to undergo before the initial quantum uncertainty in its direction is magnified into a significant uncertainty with practical consequences, say $\Delta\theta = 1\,radian$? This should suggest to you an unbeatable gambling device whose outcome is truly random.

2 HUP: Limitations on energies

A hard ball of mass M subjected to the gravitational force Mg bounces off of the floor at $z = 0$. The potential energy experienced by the ball is sketched in the figure below. Note that the ball cannot go through the floor to regions where $z < 0$. Due to the uncertainty principle, even in its minimum energy state, the ball can never be consistently at rest at $z = 0$. Instead, the ball always has some characteristic non-zero kinetic energy $T > 0$ and height $z = \Delta Z > 0$ above the ground.


(y)
(x)
D
(ΔΘ1)
(Δ Θ0)
R
(Δy)
(Pin # 1)


V(z)=Mgz
z=0
z ->

2.1 Potential Energy

Suppose that the ball is typically confined to the region between $z = 0$ and a height $z = \Delta Z$ above the ground. From this, give an approximation for the potential energy expected on average for experiments on a large number of systems. Plot the potential energy V as a function of the parameter ΔZ.

2.2 Kinetic Energy

Given ΔZ, what limitations (if any) are there on the z-component of the momentum of the ball? On the same plot as for 2.1, sketch the allowed values for the kinetic energy T of the ball as a function of ΔZ.

2.3 Minimum Total Energy

On your sketch indicate also the region of allowed *total energies* $E = T + V$ as a function of ΔZ and the point corresponding to the lowest possible average energy. Compute the value of ΔZ which allows for this minimum energy and the corresponding minimum energy E_{min}.

2.4 Energy of Confinement

Verify that, as discussed in class, the kinetic energy associated with the confinement to a region of size ΔZ is approximately $\hbar^2/(M\Delta Z^2)$.

2.5 Correspondence Principle

If $M = 100\,g$ and $g = 980\,cm/s^2$, evaluate the typical height ΔZ and minimum energy E_{min}. Why are these values so small? By taking the limits and comparing with what you expect classically, show that your results are consistent with the classical limit $\hbar \longrightarrow 0$.

3 Franck-Hertz Experiment

(Courtesy of Prof. Nihat Berker)

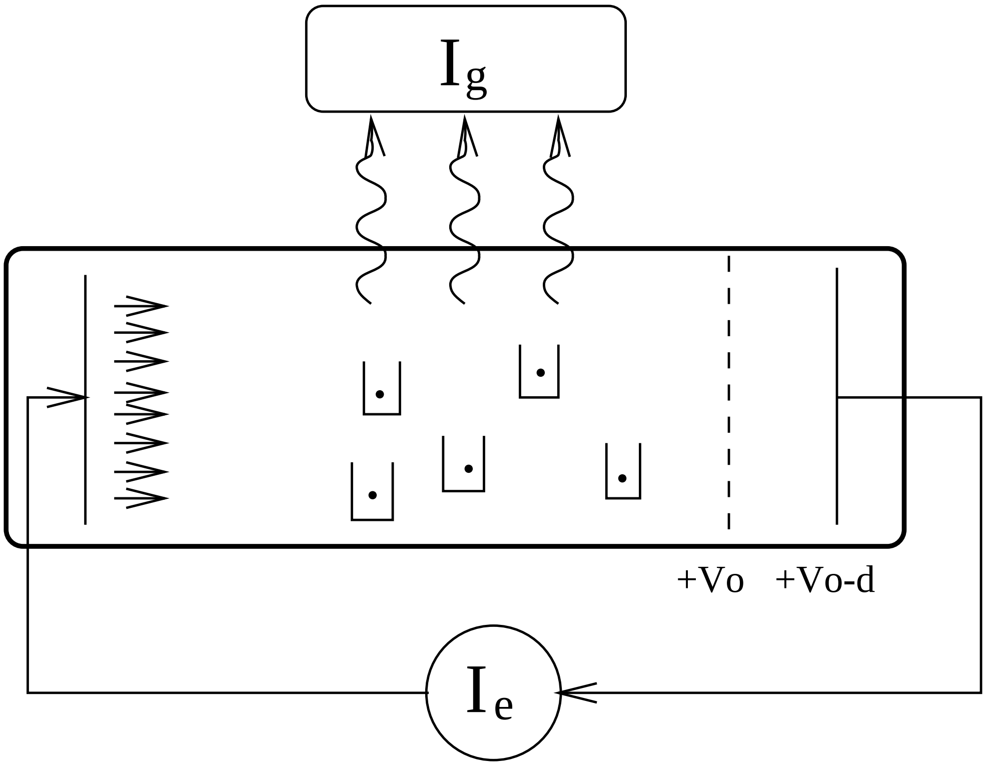
Consider doing the *Frank-Hertz* experiment as in the diagram below.

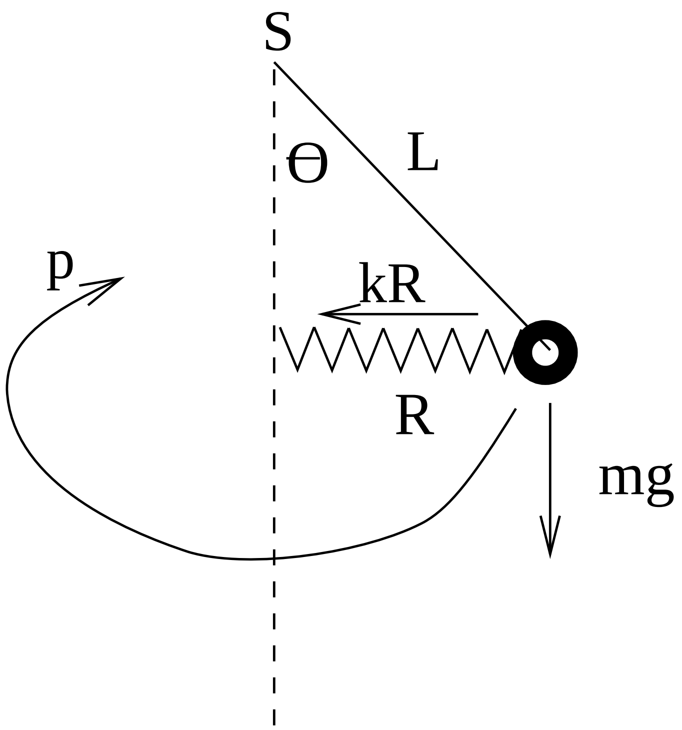
V_0 is the accelerating voltage, d is an infinitesimal retarding voltage. The experimental vessel contains "atomic" systems which can be represented by particles of mass m, each bound in an infinite square well potential of width a. (The energy levels for such square wells are derived in the class notes CN "Bohr-Sommerfeld Quantization...") I_e is the electron current and I_g is the emitted light intensity. Plot the behavior of these two quantities as V_0 is scanned from 0 to $5h^2/4ma^2e$.

4 Quantization and Classical Correspondence

In this problem, you will analyze the consequences of the de Broglie relations (i.e., Bohr quantization) on the motion of "conical pendulum" attached to a spring (see figure). This system is like a normal pendulum, there is a mass m suspended from a point S and feeling the force of gravity mg directed downward.

There are two important differences from the normal pendulum, however. First, the mass is free to move in anyway about the point S and make circular orbits of radius R as indicated in the figure. Second, the mass is not only connected to the point S with a rigid string but also to the rotation axis with with a spring with spring constant k and natural length zero. The force from the spring is thus directed radially inward with magnitude kR.


I_g
+Vo
+Vo-d
I_e


S
Θ
L
p
kR
R
mg

For reference, we will call the momentum of the particle p, the angle which the rod L makes with the direction of the vertical Θ, the distance from the mass m to the vertical line R, the total energy (kinetic plus potential) of the pendulum E, and the period when the particle undergoes circular motion T. In the Bohr analysis of the motion there must be an integral number n of de Broglie wavelengths around the circumference of the orbit of radius R. In a quantum state n, there are specific values allowed to the angle and energy of the system, which we shall call Θ_n, and E_n, respectively. Your task is to use the Bohr quantization condition to compute Θ_n and E_n, and understand the behavior of the level spacings in the spectrum E_n by using the correspondence principle.

4.1 Classical orbits

What is the momentum p of the particle as a function of Θ as the particle moves in its circular orbit? What are the period T and total energy E of this motion? Express all three answers in terms of just Θ, k, m, L and g.

4.2 Correspondence principle argument

If a small test charge q is placed on the mass m, what frequency radiation would the pendulum emit when rotating in a circular orbits at an angle Θ? Explain how this leads to the expectation that the spacing between neighboring levels obey $\Delta E \longrightarrow h/T_{classical}$, where $T_{classical}$ is the classical period of the motion and ΔE is the spacing between successive quantum states.

4.3 Semiclassical quantization

Use the Bohr quantization condition that there must be an integral number n of de Broglie wavelengths around the circumference of the orbit of radius R to find a condition on the allowed quantized values Θ_n in terms of the quantum number n and the constants $\hbar$, m, g and k and L.

4.4 Spacing between states

Under the assumption that $\hbar << \sqrt{m^2L^3g}, \sqrt{mkL^4}$ (the classical limit), give an expression for the spacing $\Delta\Theta = \Theta_n - \Theta_{n-1}$ between the allowed quantized angles of the system in terms of the classical angle of the pendulum Θ and the parameters $\hbar$, m, L, k and g.

4.5 Correspondence principle

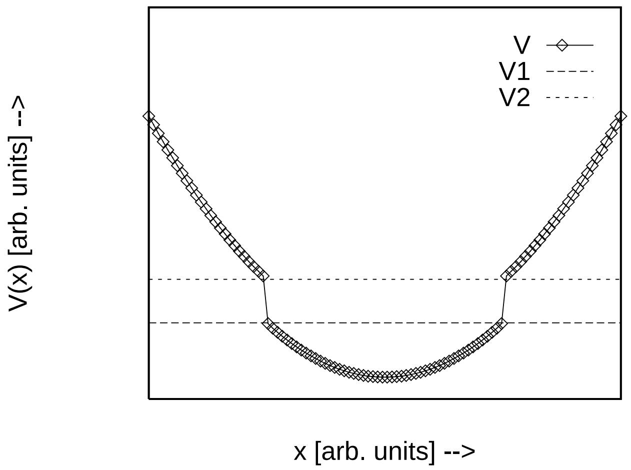
From this point on in the problem, make the simplifying assumption that $k \to 0$! Under these conditions, use your result from 4.4 to determine the spacing $\Delta E \equiv E_n - E_{n-1}$ between neighboring energy levels in terms of the angle of the pendulum Θ and verify explicitly that the behavior predicted in 4.2 is seen.

4.6 Example of exploiting the correspondence principle

Note that no computations are required for this problem.

Use the correspondence principle to sketch the energy spectrum of the allowed states E_n of a particle of mass m moving in the potential V indicated in the figure below. The potential (marked by diamonds in the figure) consists of two simple harmonic oscillator potentials of the *same* spring constant k pasted together with a discontinuity bridging the energies V_1 and V_2 indicated in the figure. The level of $V = 0$ corresponds to the minimum potential in the plot.

Be sure to include the regions $E < V_1$, $V_1 < E < V_2$ and $V_2 < E$ in your sketch. For the purposed of the sketch you may assume that there are five or so levels in each of the ranges $E < V_1$ and $V_1 < E < V_2$.


V
V1
V2
V(x) [arb. units] -->
x [arb. units] -->