MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Physics

Physics 8.04 May 18, 1996

8.04 Quiz II, Spring 1996

Instructions

1. Closed book; no notes.
2. Please start solutions on the same page on which the problem is given.
3. Use subsequent blank pages if necessary

NAME:

Recitation Instructor:

Problem	Score	Grader
1.		
2.		
3.		
4.		
Total (100 pts)		

8.04 Formula Sheet: Quiz II

Physical Constants

Quantity	Symbol	Value
Atomic mass unit	amu	$1.661 \times 10^{-24}\, g$
Electronic Mass	m_e	$0.9109 \times 10^{-27}\, g$
Electronic Charge	e	$4.803 \times 10^{-10}\, esu$
Boltzmann's Constant	k	$1.381 \times 10^{-16}\, erg/^oK$
Planck's Constant	h	$6.626 \times 10^{-27}\, erg\, s$
	$\hbar$	$1.054 \times 10^{-27}\, erg\, s$
Speed of Light	c	$2.998 \times 10^{10}\, cm/s$
Electron Volt	eV	$1.602 \times 10^{-12}\, erg$
Angstrom	Å	$10^{-8}\, cm$
Stefan's Constant	σ	$5.67 \times 10^{-5} erg/s/cm^2/^oK^4$
Wien's displacement constant		$0.2898\, cm\, ^oK$

Quantum—Classical Correspondences

Quantum Quantity	Relationship	Newtonian Quantity
Wavelength	$\lambda = h/p$	Momentum
Frequency	$\nu = E/h$	Energy
Group velocity	$c_g \equiv \frac{d\nu}{d(1/\lambda)} = v$	Velocity
Phase velocity	$c_p \equiv \lambda\nu = E/p$	???
Index of refraction		Potential
Amplitude	$\lvert A\rvert^2 = P$	Probability

Thermal Radiation Formulas

$$R_T = \sigma T^4$$

$$R_T(\nu)d\nu = \left[\frac{c}{4}\right]\left[\frac{8\pi\nu^2}{c^3}d\nu\right]\left[\frac{h\nu}{e^{\beta h\nu} - 1}\right]$$

$$R_T(\nu)d\nu = \frac{c}{4} \cdot u_T(\nu)d\nu$$

$$g(\nu)d\nu = \frac{8\pi}{c^3}\nu^2 d\nu$$

$$\mathcal{E}_T(\nu) = \frac{h\nu}{e^{h\nu/kT} - 1}$$

Heisenberg Uncertainty Principle

$$\Delta X\, \Delta P \geq \hbar/2$$

Delta Function and Fourier Transforms

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dx}{2\pi}\, e^{ikx}$$

$$\delta(k) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ikx}$$

$$\delta(u) = \lim_{\epsilon\to 0} \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + u^2}$$

$$f(x) \equiv \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \tilde{f}(k) e^{ikx}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, f(x) e^{-ikx}$$

Hermitian Inner Product and Cauchy Schwartz Inequality

$$(\psi, \phi) \equiv \int_{-\infty}^{\infty} dx\ \psi^*(x)\phi(x)$$

$$|(\psi, \phi)|^2 \le (\psi, \psi)(\phi, \phi)$$

Quantum Averages and Operators

$$< \mathcal{O} > = (\psi, \hat{\mathcal{O}}\psi)$$

$\hat{x} = x \qquad \hat{p} = \frac{\hbar}{i}\partial_x \qquad$ (position representation)

$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$

$\hat{\tilde{x}} = i\partial_k \qquad \hat{\tilde{p}} = \hbar k \qquad$ (momentum representation)

General uncertainty principle, for any two observables A and B and associated operators $\hat{A}$ and $\hat{B}$:

$$\Delta A\, \Delta B \ge \frac{1}{2}| < [\hat{A}, \hat{B}] > |$$

Time Dependent Schrödinger Equation and General Features of Its Solutions

TDSE:

$$\hat{H}\Psi(x,t) = i\hbar\partial_t\Psi(x,t).$$

Conservation of Probability:

$$\partial_t\mathcal{P}(x,t) + \partial_x j(x,t) = 0,$$

where

$$\mathcal{P}(x,t) \equiv |\Psi(x,t)|^2$$

and

$$\begin{aligned} j(x,t) &\equiv \frac{\hbar}{2mi}\left(\Psi^*(x,t)\partial_x\Psi(x,t) - \Psi(x,t)\partial_x\Psi^*(x,t)\right) \\ &= \frac{\hbar}{m}\,\mathrm{Im}\left\{\Psi^*(x,t)\partial_x\Psi(x,t)\right\}. \end{aligned}$$

Ehrenfest's Theorem:

$$\begin{aligned} \frac{d}{dt} < x > &= \frac{< p >}{m} \\ \frac{d}{dt} < p > &= < -\partial_x V(x) > . \end{aligned}$$

General Ehrenfest's theorem for any observable $\mathcal{O}$ and associated operator $\hat{\mathcal{O}}$:

$$\frac{d}{dt} < \mathcal{O} > = < \frac{i}{\hbar}[\hat{H},\hat{\mathcal{O}}] > .$$

Time *In*dependent Schrödinger Equation and General Features of Its Solutions

TISE:

$$\hat{H}\psi_n(x) = E_n\psi_n(x).$$

Orthogonality among states:

$$(\psi_n,\psi_m) = \delta_{nm} \equiv \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} .$$

Boundary conditions:

$$\begin{cases} 1) & \psi_n(x) \text{ must be normalizable} \\ 2) & \text{So long as } V(x) \text{ is finite, } \psi_n(x) \text{ and } \psi_n'(x) \text{ must be continuous across all boundaries} \end{cases} .$$

Quiz Problems:

Those sections marked with an asterisk (*) require some additional time and/or thought. You may wish to skip those, using the results given, and return to them after you complete the rest of the exam.

Contents

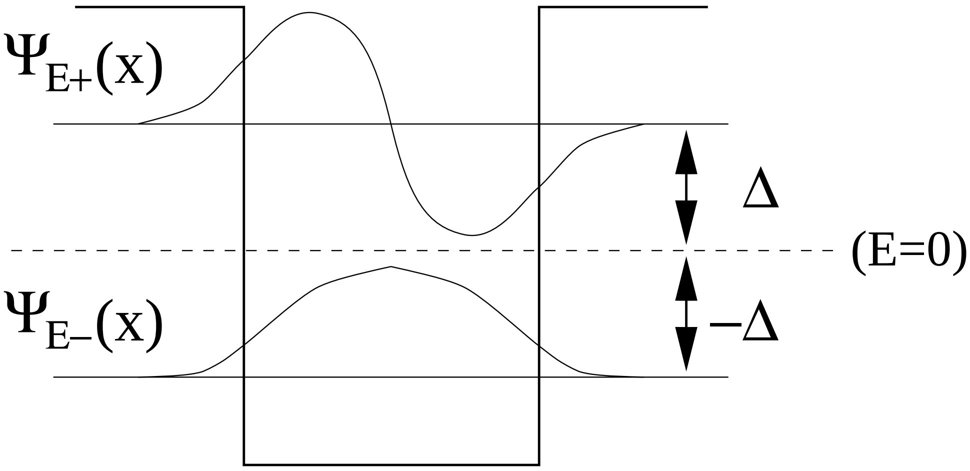

Figure 1: Potential Well Binding Two States

Observed Value	Wavefunction of Pure State (properly normalized)	$P(E=\Delta)$	$P(E=-\Delta)$
$E=\Delta$	$\Psi_{E_+}(x)$		
$E=-\Delta$	$\Psi_{E_-}(x)$		
$F=\Delta$	$\Psi_{F_+}(x)=\frac{1}{\sqrt{2}}(\Psi_{E_+}(x)+\Psi_{E_-}(x))$		
$F=-\Delta$	$\Psi_{F_-}(x)=\frac{1}{\sqrt{2}}(\Psi_{E_+}(x)-\Psi_{E_-}(x))$		
$G=\Delta$	$\Psi_{G_+}(x)=\frac{1}{\sqrt{2}}(\Psi_{E_+}(x)+i\Psi_{E_-}(x))$		
$G=-\Delta$	$\Psi_{G_-}(x)=\frac{1}{\sqrt{2}}(\Psi_{E_+}(x)-i\Psi_{E_-}(x))$		

Table 1: Table of pure states of the new observables F and G

1 Representations (20 pts total)

Background: In this problem we will consider the physics of particles which remain bound to the well in Figure 1, which binds only two pure states of energy, $\Psi_{E_+}(x)$ and $\Psi_{E_-}(x)$. The zero of energy is chosen so that the energies of these two states are Δ and $-\Delta$, respectively.

In addition to the energy E, we are told that there are two other physical observables for particles bound to the well, which we will call F and G. Measurements of these observables also always only produce the two values $\pm\Delta$ for particles bound in the well. The pure states associated with these values of F and G are given in terms of the pure states of energy in Table 1. Given what you have learned in 8.04, you are able to make predictions on the probability distribution for measurements on G given *only* information about the distributions for measurements of E and F!

1.1 Probabilities in the E representation (5 pts)

Energy measurements are performed on systems in each of the six states in Table 1. Use the *Principle of Quantum Superposition* to complete the table by giving the probabilities of finding

$E = \Delta$ and $E = -\Delta$ in measurements performed on systems in each of the six states.

1.2 G representation (5 pts)

Show how how the state Ψ_{E_+} may be written as a superposition of the two pure states of G.

For a particle in the state Ψ_{E_+}, what is the probability of finding the value $+\Delta$ in an experiment measuring the observable G?

1.3 (*) Specification of the state from experimental information (5 pts)

Note that by using the same procedure as in the previous problem, you may also derive the following superpositions. (**You need not derive these on the exam, you may take them as given.**)

$$\begin{aligned}
\Psi_{E_+} &= \frac{1}{\sqrt{2}}\left(\Psi_{F_+} + \Psi_{F_-}\right) \\
\Psi_{E_-} &= \frac{1}{\sqrt{2}}\left(\Psi_{F_+} - \Psi_{F_-}\right) \\
\Psi_{E_-} &= \frac{1}{i\sqrt{2}}\left(\Psi_{G_+} - \Psi_{G_-}\right)
\end{aligned}$$

Now, you are given a new state $\Phi(x) = c_+\Psi_{E_+}(x) + c_-\Psi_{\phi_-}(x)$. Repeated measurements yield that $P(E = \Delta) = 1/2$ and $P(E = -\Delta) = 1/2$, but $P(F = \Delta) = f$ and $P(F = -\Delta) = 1 - f$. Use this information to show 1) that c_+ and c_- must have the form

$$\begin{aligned}
c_+ &= \frac{1}{\sqrt{2}}e^{i\phi_+} \\
c_- &= \frac{1}{\sqrt{2}}e^{i\phi_-},
\end{aligned}$$

and 2) that

$$\cos(\phi_+ - \phi_-) = 2f - 1.$$

Hints: 1) Write Φ as a superposition of Ψ_{F_+} and Ψ_{F_-}. 2) Use the fact that $|z|^2 = z^* z$.

1.4 Physical Prediction (5 pts)

Given systems in the state $\Phi(x)$ from 1.3, show that the probability $P(G = \Delta)$ is given by

$$P(G = \Delta) = \frac{1}{2}\left(1 - \sin(\phi_+ - \phi_-)\right).$$

2 Operators (25 pts total)

Background: In this problem you will consider a particle of mass m subjected to the potential of a Simple Harmonic Oscillator (SHO) of frequency ω, with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2.$$

Many general facts about the states of the SHO may be derived without knowing the states – just by using operators.

In this problem you will will make use of the following two operators,

$$\begin{aligned}
\hat{R} &\equiv \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_o} - x_o\frac{i}{\hbar}\hat{p}\right) \\
\hat{L} &\equiv \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_o} + x_o\frac{i}{\hbar}\hat{p}\right),
\end{aligned}$$

with x_o defined as

$$x_o \equiv \sqrt{\frac{\hbar}{m\omega}}.$$

2.1 Basic Properties of the operators R and L (5 pts)

<u>Compute</u> both $\hat{R}\left(\hat{L}\psi(x)\right)$ and $\hat{L}\left(\hat{R}\psi(x)\right)$.

Math tip: After doing $\hat{R}\hat{L}\psi(x)$ first, you are permitted to just write down the answer for $\hat{L}\hat{R}\psi(x)$ after carefully considering the sign changes.

2.2 Commutator (5 pts)

<u>Show</u> $[\hat{L}, \hat{R}] = \hat{1}$. In other words, $[\hat{L}, \hat{R}]\psi(x) = \psi(x)$, for any state $\psi(x)$.

Hint: This provides a quick check of your result in the previous problem. After doing (2.1), this should require little additional algebra.

2.3 Rewriting the Hamiltonian (5 pts)

<u>Show</u> that $\hat{H} = \hbar\omega(\hat{R}\hat{L} + 1/2)$, in other words that $\hat{H}\psi(x) = \hbar\omega(\hat{R}\hat{L} + 1/2)\psi(x)$.

Hint: After doing (2.1), this should require little additional algebra.

2.4 A very simple equation for the ground state (5 pts)

Prove that if $\hat{L}\psi_0(x) = 0$, then $\psi_0(x)$ is a pure state of energy and give the associated value of the energy.

Hint: You should be able to do this problem using the results given to you in (2.2) and/or (2.3) without much additional algebra.

2.5 (*) First excited state (5 pts)

Show that, if $\psi_0(x)$ is the same state from the previous problem obeying $\hat{L}\psi_0(x) = 0$, and if we define a new state $\psi_1(x) \equiv \hat{R}\psi_0(x)$, then $\psi_1(x)$ is also a pure state of $\hat{H}$, and give the energy of the new state $\psi_1(x)$.

Hint: You should be able to do this problem using the results given to you in (2.2-2.4) without much additional algebra.

3 Averages and Solutions of the TDSE (30 pts total)

Background: Consider a particle of mass m subjected to the potential of a Simple Harmonic Oscillator (SHO) of frequency ω. The Hamiltonian for this system is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2.$$

The allowed energies and several wavefunctions of the SHO are

$$\left\{ \begin{array}{cc} E_0 = \frac{1}{2}\hbar\omega & \psi_0(x) = \frac{1}{\pi^{1/4}x_0^{1/2}} e^{-(x/x_0)^2/2} \\ E_1 = \frac{5}{2}\hbar\omega & \psi_1(x) = \frac{\sqrt{2}}{\pi^{1/4}x_0^{1/2}}(x/x_0)e^{-(x/x_0)^2/2} \\ E_2 = \frac{5}{2}\hbar\omega & \psi_2(x) = \frac{\sqrt{2}}{\pi^{1/4}x_0^{1/2}}\left((x/x_0)^2 - 1/2\right)e^{-(x/x_0)^2/2} \\ \vdots & \vdots \\ E_n = (n+1/2)\hbar\omega & \frac{1}{2^n n! \pi^{1/4} x_0^{1/2}}\mathcal{H}_n(x/x_0)e^{-(x/x_0)^2/2} \\ \vdots & \vdots \end{array} \right\},$$

where, as before,

$$x_o \equiv \sqrt{\frac{\hbar}{m\omega}},$$

and $\mathcal{H}_n(x)$ happens to be the n^{th} Hermite polynomial. (**Note:** this problem requires *no* knowledge of Hermite polynomials.)

3.1 Time evolution (7 pts)

Suppose that at time $t = 0$, the state of the system (properly normalized) is given by $\Phi(x, t = 0) = \frac{1}{\sqrt{2}}(\psi_0(x) + \psi_1(x))$. Write down the state of the system $\Phi(x, t)$, if undisturbed, at times $t > 0$.

3.2 (*) Center of the packet (8 pts)

What is the average value of $< x >$ you would expect to find in measurements carried out at time t? Leave your result in terms of m, $\hbar$, ω and t. Any integrals which you know to be zero by symmetry you may eliminate without justification – just be sure.
Hint: You will be best off by writing your expression as a sum of integrals where you substitute in

the explicit forms for ψ_0 and ψ_1 at the *last possible moment.* You may use the following standard integral:

$$\int_{-\infty}^{\infty} u^n e^{-u^2}\, du = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \sqrt{\pi}\frac{(n-1)!!}{2^{n/2}} \equiv \sqrt{\pi}\frac{n!}{2^n\,(n/2)!} & \text{if } n \text{ is even} \end{cases}$$

3.3 General behavior of averages (7 pts)

Given $\Psi(x, t = 0) \equiv \sum_n c_n \psi_n(x)$, compute the average value for any observable O and time t, $< O >_t \equiv (\Psi(x,t), \hat{O}\Psi(x,t))$, in terms of t, ω, c_n, and $(\psi_n, \hat{O}\psi_m)$.

3.4 Periodicity of averages in the SHO (8 pts)

For your result in the previous section, show that the average for any observable repeats periodically in time with the classical period of the oscillator. In other words, that $< O >_{t+2\pi/\omega} = < O >_t$.

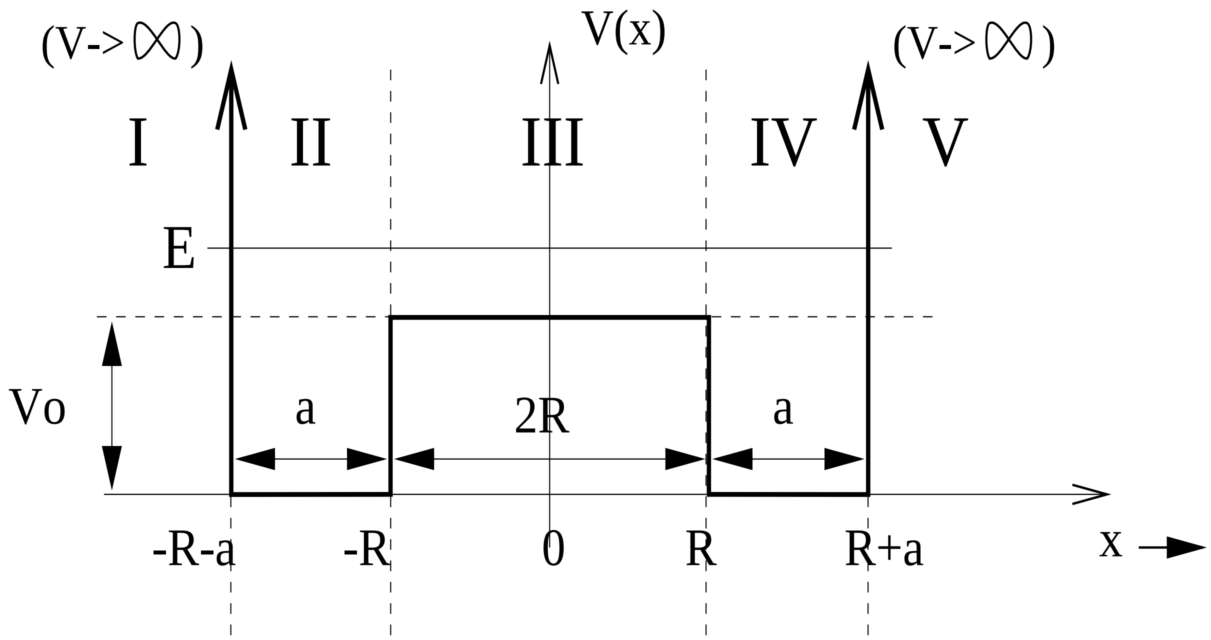

Figure 2: Double Potential Well

4 Solutions of the TISE (25 pts total)

Background: A particle of mass m is confined in one dimension to the "double well" potential indicated in Figure 2. Interestingly, for this well there are special values of V_o for which the ground state energy does not depend on the size of the bump in the middle as long as we keep a constant. The goal of this problem is to determine for what *finite non-zero* values of V_o is the ground state energy independent of R, where $a > 0$.

4.1 Wavefunctions (8 pts)

Write down the most general form for *even* wavefunctions in this well consistent with 1) the Time Independent Schrödinger Equation (TISE), 2) the symmetry of the ground state, and 3) the boundary conditions at $x = -R - a$ and $x = R + a$. Be sure to specifiy your wavefunction in each of the regions I-V, and give explicit formulae in terms of m, $\hbar$, V_o, and E for any wave vector constants (k's or κ's) which you introduce in your functions.

4.2 Condition on Allowed Energy States (9 pts)

Give a transcendental equation which determines the allowed energies E for *even* states.

4.3 (*) Ground state energy independent of R (8 pts)

From your allowed energy condition, determine the value V_o for which the ground state energy stays constant as R varies. Sketch your ground state wavefunction for this value of V_o.