Massachusetts Institute of Technology

Department of Physics

Physics 8.04 March 13, 1997

Quantum Physics I, Spring 1997

Problem Set 5

(Due Friday, March 21 at 5:00 pm sharp.)

Agenda and **required readings** for the weeks of March 17:

The abbreviations are RH = *Resnick and Halliday* (section numbers), G = *Gasiorowicz* (section(s) contained within the given range of pages), H = *Heisenberg* (section numbers), FT = *French and Taylor* (section numbers), CN = class notes provided on the web.

- L11, 03/18: Normalization, Parseval's Theorem, Hermitian Inner Product; **Averages and Operators**, Pure States and Operators
  Readings: CN "Quantum Averages and Operators", G 46-48, 55-64
- L12, 03/20: Quantum Dynamics, Movies of Wave Functions, Proof of $F = ma$, Conservation Laws
  Readings: RH CN "General Features of the Time Dependent Schrödinger Equation (TDSE)," G 41-45, 125-128

Work requested of the students is highlighted by either a question mark (?) or underlining.

Contents

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1 Derivation of Operators from the de Broglie Hypothesis (25 pts)

1.1 Completing the basic set from lecture

Background: In lecture, we found representations for the operators for x and p in the position representation ($\hat{x} \to x$ and $\hat{p} \to \frac{\hbar}{i}\partial_x$, respectively), and for the momentum operator in the position representation $\hat{p} \to p$.
Complete the set by deriving the representation for the position operator $\hat{x}$ in the *momentum* representation.

1.2 Powers of x and p

Beginning with the definitions of the operators $\hat{x^n}$ and $\hat{p^n}$,

$$< x^n >= \int dx\, \psi^*(x) x^n \psi(x) \equiv < \psi|\hat{x^n}|\psi >$$

$$< p^n >= \int dp\, \tilde{\psi}^*(p) p^n \tilde{\psi}(p) \equiv < \psi|\hat{p^n}|\psi >,$$

derive position and momentum space representations for $\hat{p^n}$ and $\hat{x^n}$.

1.3 Arbitrary functions of x and p

Show that the effect of the "infinitesimal translation" operator $\hat{T}_\epsilon \equiv (1 - i\epsilon\hat{p}/\hbar)$ on the wave function $\psi(x)$ is to translate it forward in space by a small distance ϵ,

$$\hat{T}_\epsilon \psi(x) \approx \psi(x - \epsilon).$$

Using the idea that any finite distance a may be may up of N tiny steps of size ϵ ($a = N\epsilon$ for $N \to \infty$ and $\epsilon \to 0$), derive the form of the finite translation operator $\hat{T}_a$ which shifts a the wave function by the finite distance a.
Hint: Recall $\lim_{N\to\infty}(1 + x/N)^N = e^x$.

Now, using the expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

the Taylor series

$$f(x+c) = \sum_{n=0}^{\infty} \frac{c^n}{n!} f^{[n]}(x),$$

where $f^{[n]}$ denotes the n^{th} derivative of $f(x)$, and your result for 1.2, verify that for your translation operator indeed

$$\hat{T}_a \psi(x) = \psi(x-a).$$

1.4 Beyond one dimension

Taking the de Broglie hypothesis in three dimensions to be that $e^{i\vec{k}\cdot\vec{x}}$ represents a pure state of momentum $\vec{p} = \hbar\vec{k}$, derive representations for the *three dimensional* momentum and position operators $\hat{\vec{p}}$ and $\hat{\vec{x}}$ in *both* the position and momentum representations.

1.5 Hamiltonian operators

Derive, in the position representation, the Hamilton $\hat{H}$ (total energy) operator for a single particle moving with energy $E = \frac{\vec{p}^2}{2m} + V(\vec{x})$ from the condition that the average energy of the particle computed from $< \psi|\hat{H}|\psi >$ must match the average energy computed from the position and momentum operators $\hat{\vec{x}}$ and $\hat{\vec{p}}$.

2 Application of Operators (25 pts)

Parts 2.2-2.5 below refer to the quantum state which may be described by the momentum space wave function

$$\tilde{\Psi}(k) \quad = \quad A e^{-D|k|} e^{ik(x-x_o)},$$

where A is a normalization constant.
You may find the following derivatives and integrals useful:

$$\begin{aligned}
\frac{d}{du}\left(\frac{1}{a^2+u^2}\right) &= -\frac{2\,u}{\left(a^2+u^2\right)^2} \\
\frac{d^2}{du^2}\left(\frac{1}{a^2+u^2}\right) &= \frac{6\,u^2 - 2\,a^2}{\left(a^2+u^2\right)^3} \\
\int_{-\infty}^{\infty} \frac{u^2}{\left((u-b)^2+a^2\right)^2} du &= \frac{\pi}{2}\frac{\left(a^2+b^2\right)}{a^3} \\
\int_{-\infty}^{\infty} u^2 e^{-2\,a|u|} du &= \frac{1}{2a^3}
\end{aligned}$$

2.1 Normalization

Normalize the wave function $\tilde{\Psi}(k)$.

2.2 Probability distributions

What is $\mathcal{P}(x)$, the probability distribution for measurements of the position of particles in the state $\tilde{\Psi}(k)$?
What is $\tilde{\mathcal{P}}(k)$, the probability distribution for measurements of the momentum of particles in the state $\tilde{\Psi}(k)$?

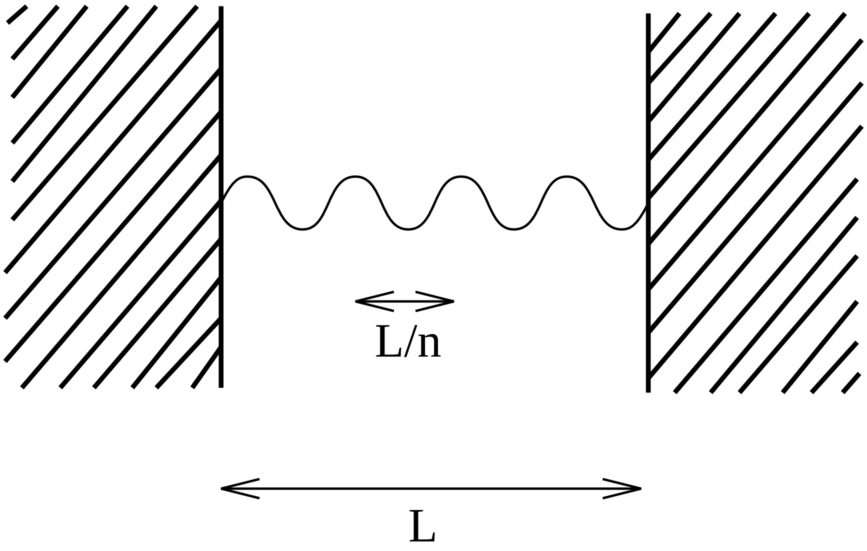

Figure 1: Particle trapped between two plates

2.3 Expectations

What average $< x >$ position do you expect to find in measurements of the location of the particle? What average momentum $< p >$ do you expect to find in measurements of the momentum of the particle?

2.4 Uncertainties

What is the uncertainty Δx in position measurements carried out on this state?
What is the uncertainty Δp in momentum measurements carried out on this state?
Show explicitly that these uncertainties are consistent with the Heisenberg Uncertainty Principle.

2.5 Pure states of energy

Determine (up to an arbitrary choice of the zero of potential energy), a potential $V(x)$ for which $\tilde{\Psi}(k)$ is a pure state of the Hamiltonian.

3 Particle Trapped in a Cavity Revisited (25 pts)

Consider a particle, as discussed in lecture, which is trapped between two plates as indicated in Figure 1 and described by a standing wave pattern of wavelength $\lambda = L/n$ for a relatively large value of n, say $n = 100$. Note that "trapped" means that the probability of finding the particle outside of the region $0 < x < L$ is zero.

3.1 Wave function

Write down a *continuous* normalized wave function $\psi(x)$ which corresponds to this state.
Hint: Keep in mind that $\psi(x) = 0$ for $x \leq 0$ and $x \geq L$.

3.2 Momentum Distribution

Determine and sketch the probability distribution $\tilde{\mathcal{P}}(k)$ for the outcomes of measurement of the momenta of the particle in this state. Explain, physically (using the de Broglie Hypothesis) the approximate value of the momentum associated with the main peak(s) in the distribution.

3.3 Heisenberg Uncertainty Principle in Action

Explain why, in this case, the peak(s) of the distribution are not infinitely narrow, and show qualitatively that their width is in accord with the Heisenberg Uncertainty Principle.

4 Uncertainty Principle (25 pts)

Background: In this problem you will prove that the Heisenberg Uncertainty Principle (HUP) follows directly from the operator formalism which we developed from the de Broglie hypothesis!!! You may regard this proof as either one confirmation that our formalism is consistent with our experimental observations of the HUP or as the precise mathematical link between the Davisson-Germer/G.P. Thomson observations that $\lambda = h/p$ and the HUP statement that $\Delta x\, \Delta p_x \geq \hbar/2$.

Hint: You may wish to consult *Gasiorowicz*, pp. 437-438. Also, you may consult Heisenberg's own proof in his book on pp. 15-18.

4.1 Shift away the mean values

Let $\hat{\rho}$ and $\hat{\chi}$ be two new operators defined as $\hat{\rho} \equiv \hat{p} - <\hat{p}>$ and $\hat{\chi} \equiv \hat{x} - <\hat{x}>$. Show

$$(\Delta p)^2 = (\hat{\rho}\psi, \hat{\rho}\psi),$$

and

$$(\Delta x)^2 = (\hat{\chi}\psi, \hat{\chi}\psi).$$

4.2 Generate an inequality

Use the Schwartz Inequality from the Problem Set 4 with $f = \hat{\rho}\psi$ and $g = \hat{\chi}\psi$ to show

$$(\Delta x)^2(\Delta p)^2 \geq |<\hat{\rho}\hat{\chi}>|^2.$$

4.3 Separation of the terms

Show that

$$\hat{\rho}\hat{\chi} = \frac{1}{2}\{\hat{\rho}, \hat{\chi}\} + \frac{1}{2}[\hat{\rho}, \hat{\chi}],$$

where $[\hat{x}, \hat{p}] \equiv \hat{x}\hat{p} - \hat{p}\hat{x}$ and $\{\hat{x}, \hat{p}\} \equiv \hat{x}\hat{p} + \hat{p}\hat{x}$.
Next, use the properties of the Hermitian inner product to show that if we then take $<\{\hat{\rho}, \hat{\chi}\}>$, we get a purely real number, whereas if we take $<[\hat{\rho}, \hat{\chi}]>$ we get a purely imaginary number. *Do not use any* specific *properties of the $\hat{x}$ or $\hat{p}$ operators in this step!*

4.4 Two general uncertainty principles

From your results show both,

$$\begin{aligned}\Delta x\,\Delta p &\geq \frac{1}{2}|<[\hat{x},\hat{p}]>| \\ \Delta x\,\Delta p &\geq \frac{1}{2}|<\{\hat{x},\hat{p}\}>|,\end{aligned}$$

where $[\hat{x},\hat{p}] \equiv \hat{x}\hat{p} - \hat{p}\hat{x}$ and $\{\hat{x},\hat{p}\} \equiv \hat{x}\hat{p} + \hat{p}\hat{x}$.
At this point analyze your proof in steps 4.1 through 4.4 critically for dependencies on the specific properties of $\hat{x}$ and $\hat{p}$ and give the most general statement(s) with which you are comfortable for the spread in results $\Delta A\,\Delta B$ of the measurements of an arbitrary pair of observables A and B.

4.5 Application to x and p

Compute $[\hat{x},\hat{p}]$ and $\{\hat{x},\hat{p}\}$, and use whichever form is appropriate to complete the proof of the specific form of Heisenberg Uncertainty Principle (HUP) stated in the first phase of the course,

$$\Delta x\,\Delta p \geq \hbar/2.$$