

CORNELL UNIVERSITY
Department of Physics

Physics 214 Prelim I Fall 2001

NAME: SOLUTIONS

SECTION: _____

Instructions

- To receive credit, you must place your answers in the boxes provided whenever required.
- Closed book; no notes. You may use a calculator.
- Check that you have all **16** pages (including cover page). The formula sheet is distributed separately.
- **Important note:** Except for some challenge problems, each part of this exam is designed to be answered without the answers of previous parts. The parts within a given problem become more and more difficult. If you get stuck on one part, skip to the next problem and come back later if you have more time.

Problem	Score	Grader
1. (28 pts)		
2. (8 pts)		
3. (8 pts)		
4. (14 pts)		
5. (21 pts)		
6. (7 pts)		
7. (14 pts)		
Total (100 pts)		

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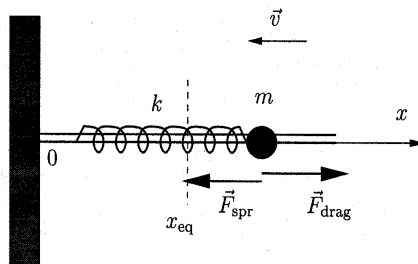


Figure 1: A mass-spring realization of a damped harmonic oscillator.

1 Problem 1: Damped Harmonic Oscillator [28 points]

A damped oscillator is modeled as a mass m with equilibrium position $x = x_{eq}$ acted on by an ideal spring of spring constant k and a viscous drag force proportional to the velocity, $\vec{F}_{drag} = -b\vec{v}$. (See Figure 1.)

(a) Equation of Motion (7 points)

Show that the Equation of Motion for the damped harmonic oscillator is:

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega_0^2(x - x_{eq}) = 0, \quad \omega_0 \equiv \sqrt{\frac{k}{m}}. \quad (1.1)$$

Writing $\vec{F}_{net} = m\vec{a}$ for the damped harmonic oscillator we get:

$$F_{net,x} = ma_x$$

$$F_{spr,x} + F_{drag,x} = ma_x$$

$$-k(x - x_{eq}) - bmv_x = ma_x \quad | :m$$

$$-\frac{k}{m}(x - x_{eq}) - bv_x = a_x,$$

$$\text{with } k/m \equiv \omega_0^2.$$

Finally, expressing v_x, a_x in terms of the D. of F. $x(t)$, we obtain eq. (1.1):

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega_0^2(x - x_{eq}) = 0$$

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(b) Real Part of \underline{A} (7 points)

In the case $b < 2\omega_0$ the expression

$$x(t) = x_{eq} + \Re[A e^{(-b/2 + i\omega')t}], \quad \omega' \equiv \sqrt{\omega_0^2 - b^2/4} \quad (1.2)$$

is a general solution to the Equation of Motion (1.1).

Find the real part $\Re[A]$, given the initial conditions $x_0 = x(t=0)$ and $v_0 = \frac{dx}{dt}(t=0)$. Express your answer using no quantities other than x_0 , v_0 , x_{eq} , b , and ω_0 .

Imposing the initial condition $x_0 = x(t=0)$ on the general solution (1.2), we get:

$$x_0 = x(t=0) = x_{eq} + \Re[A e^{(-b/2 + i\omega') \cdot 0}] = x_{eq} + \Re[A]$$

$$\Rightarrow \boxed{\Re[A] = x_0 - x_{eq}}$$

(c) Imaginary Part of \underline{A} (7 points) Challenge Problem!

Using the same initial conditions as in part (b), find the imaginary part $\Im[A]$. Express your answer using no quantities other than x_0 , v_0 , x_{eq} , b , and ω_0 .

Hint: Write $\underline{A} = A_r + iA_i$, where $A_r = \Re[A]$ and $A_i = \Im[A]$, and solve for A_i .

Imposing the initial condition $v_0 = \frac{dx}{dt}(t=0)$ on the general solution (1.2) and writing $\underline{A} = A_r + iA_i$:

$$\begin{aligned} v_0 &= \frac{dx}{dt}(t=0) = 0 + \frac{d}{dt} \Re[A e^{(-b/2 + i\omega')t}]_{t=0} \\ &= \Re[A (-\frac{b}{2} + i\omega') e^{(-b/2 + i\omega') \cdot 0}] = \Re[(A_r + iA_i)(-\frac{b}{2} + i\omega')] \\ &= \Re[(-\frac{b}{2}A_r - \omega'A_i) + i(-\frac{b}{2}A_i + \omega'A_r)] = -\frac{bA_r}{2} - \omega'A_i \end{aligned}$$

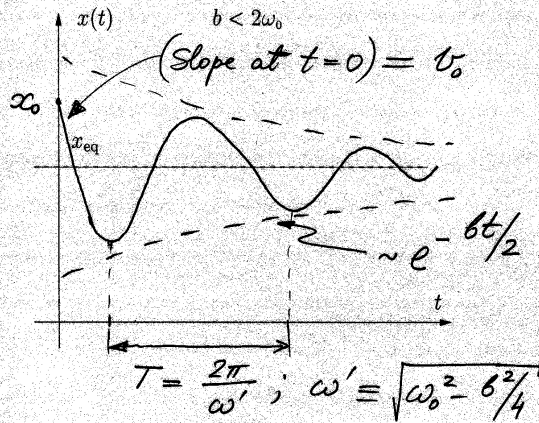
$$\Rightarrow A_i = \frac{v_0}{\omega'} - \frac{bA_r}{2\omega'}. \quad \text{Using } A_r = \Re[A] = x_0 - x_{eq} \text{ from (b) and } \omega' \equiv \sqrt{\omega_0^2 - b^2/4}, \text{ we obtain:}$$

$$\boxed{A_i = \Im[A] = \frac{1}{\sqrt{\omega_0^2 - b^2/4}} \left[v_0 - \frac{b}{2}(x_0 - x_{eq}) \right]}$$

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(d) Rough Sketch of the Solution (7 points)

Roughly sketch the solution (1.2) assuming $x_0 > x_{eq}$ and $v_0 < 0$. (Pay attention to the slope and the value of the plot at $t=0$.)



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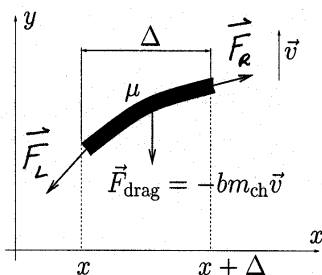


Figure 2: A chunk of string with air resistance.

2 Problem 2: Wave Equation for a String [8 points]

Note: In this problem we use the same notation as in lecture.

Consider a chunk of string of length Δ and mass per unit length μ . The string is under tension τ . The displacement in the vertical direction is given by the variable y which is a function of both position x and time t . As opposed to the derivation in lecture, where air resistance is neglected, we now include it. Note that air resistance on a chunk of string moving with velocity \vec{v} creates a force opposing the direction of the velocity, i.e., with a y -component

$$F_{\text{drag},y} = -bm_{\text{ch}}v_y, \quad (2.1)$$

where b is a positive constant and m_{ch} is the mass of the chunk. A free body diagram on Figure 2 indicates all the forces acting on the chunk. Note that the positive y direction is upward and that we ignore the (small) effects of gravity in this problem.

Which of the following formulas best approximates the wave equation for small amplitude waves on the string with air resistance? (Provide your answer in the box on the next page.)

- (A) $\mu \frac{\partial^2 y(x,t)}{\partial x^2} - b \frac{\partial y(x,t)}{\partial t} = \tau \frac{\partial^2 y(x,t)}{\partial t^2}$
- (B) $\tau \frac{\partial^2 y(x,t)}{\partial x^2} = \mu \frac{\partial^2 y(x,t)}{\partial t^2}$
- (C) $\tau \frac{\partial^2 y(x,t)}{\partial x^2} - b\mu \frac{\partial y(x,t)}{\partial t} = \mu \frac{\partial^2 y(x,t)}{\partial t^2}$
- (D) $\tau \frac{\partial^2 y(x,t)}{\partial x^2} - b\mu \frac{\partial y(x,t)}{\partial x} = \mu \frac{\partial^2 y(x,t)}{\partial t^2}$
- (E) $\tau \frac{\partial^2 y(x,t)}{\partial x^2} - b \frac{\partial y(x,t)}{\partial t} = \mu \frac{\partial^2 y(x,t)}{\partial t^2}$
- (F) $\tau \frac{\partial y(x,t)}{\partial x} - b\mu \frac{\partial^2 y(x,t)}{\partial t^2} = \mu \frac{\partial y(x,t)}{\partial t}$

As argued in lecture, the y -components of the forces \vec{F}_L and \vec{F}_R , with which the remaining string pulls the LEFT and the RIGHT end of the chunk respectively, are:

$$F_{L,y} = -\tau \frac{\partial y}{\partial x}(x,t)$$

$$F_{R,y} = +\tau \frac{\partial y}{\partial x}(x+\Delta,t)$$

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Writing $\vec{F}_{\text{net}} = m_{\text{ch}} \vec{a}$ for the chunk, we obtain:

$$F_{\text{net},y} = m_{\text{ch}} a_y$$

$$F_{L,y} + F_{R,y} + F_{\text{drag},y} = m_{\text{ch}} a_y$$

$$-\tau \frac{\partial y}{\partial x}(x,t) + \tau \frac{\partial y}{\partial x}(x+\Delta,t) - b m_{\text{ch}} v_y = m_{\text{ch}} a_y$$

Writing $m_{\text{ch}} = \mu \Delta$, we get:

$$\tau \left[\frac{\partial y}{\partial x}(x+\Delta,t) - \frac{\partial y}{\partial x}(x,t) \right] - b \mu \Delta v_y = \mu \Delta a_y \quad | : \Delta$$

$$\tau \frac{\frac{\partial y}{\partial x}(x+\Delta,t) - \frac{\partial y}{\partial x}(x,t)}{\Delta} - b \mu v_y = \mu a_y$$

Taking the limit $\Delta \rightarrow 0$ and writing v_y and a_y in terms of the D. of y . C

($v_y = \frac{\partial y}{\partial t}(x,t)$, $a_y = \frac{\partial^2 y}{\partial t^2}(x,t)$), we obtain:

$$\tau \frac{\partial^2 y}{\partial x^2}(x,t) - b \mu \frac{\partial y}{\partial t}(x,t) = \mu \frac{\partial^2 y}{\partial t^2}(x,t)$$

Obviously, this is the Wave Equation of answer C.

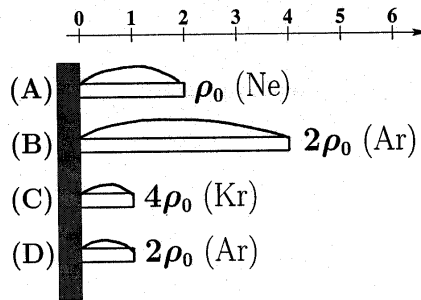


Figure 3: Closed tubes filled with noble gases.

3 Problem 3: Standing Waves in Sound Tubes [8 points]

Four tubes, each with two closed ends, are filled with different noble gases at atmospheric pressure and room temperature; therefore, they have the same bulk modulus B but different mass densities $\rho_0, 2\rho_0$, etc. (see Figure 3). The tubes also have different lengths, as shown on Figure 3. **Which two of these (A & B or B & D, etc.) tubes will have the same lowest resonant frequency? Provide your answer in the box below.**

First, relate the resonant frequency f to the wavelength λ of the standing wave formed in a tube:

$$f = \frac{v}{\lambda} \quad (\text{dispersion relation})$$

Second, since we are looking for the LOWEST resonant frequency, we have to use the wavelength corresponding to the LOWEST standing wave mode:

$$\frac{\lambda}{2} = l \Rightarrow \lambda = 2l$$

Finally, recalling that the speed of sound waves is related to the bulk modulus B and the gas density ρ by:

$$v = \sqrt{\frac{B}{\rho}}$$

we get:

$$f = \frac{1}{2l} \sqrt{\frac{B}{\rho}}$$

We thus see that tubes (A) (C) will have the same lowest resonant frequency.

A & C

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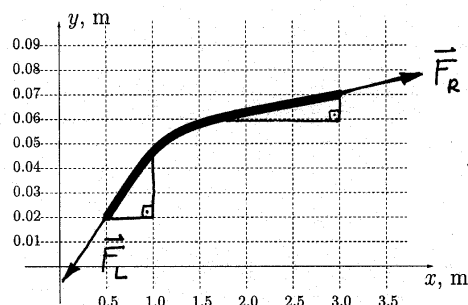


Figure 4: A chunk of vibrating string.

4 Problem 4: Forces, Slopes and Curvature [14 points]

A chunk of vibrating string (tension $\tau = 1000$ N and linear mass density $\mu = 0.6$ kg/m) is shown on the graph below: Neglecting air resistance and gravity, and using the graphical information on Figure 4, *complete the tasks below*:

(a) Forces (7 points)

For each of the parts below choose the value, (A), (B), (C), (D), (E), or (F), which is closest to the answer of the question: (Note that your answer may not match any of these exactly.)

(A) 60 N, (B) 6 N, (C) 20 N, (D) 15 N, (E) 1000 N, (F) 53 N.

Provide your answers in the boxes below each question.

- What is the magnitude of the y -component of the force on the right hand end of the chunk?

Reading the slope off the graph, we find:

$$F_{R,y} = \tau \frac{\partial y}{\partial x}(3.5\text{m}) \approx 1000\text{N} \times \left(\frac{0.01\text{m}}{1.5\text{m}}\right) \approx \underline{6.67\text{N}}$$

\Rightarrow The CLOSEST answer is (B) 6 N

B

- What is the magnitude of the y -component of force on the left hand end of the chunk?

Reading the slope off the graph, we find:

$$F_{L,y} = -\tau \frac{\partial y}{\partial x}(0.5\text{m}) \approx -1000\text{N} \times \left(\frac{0.03\text{m}}{0.5\text{m}}\right) \approx \underline{\underline{-60\text{N}}}$$

\Rightarrow For the MAGNITUDE $|F_{L,y}|$ we ought to choose answer (A) 60N.

A

- What is the magnitude of the net force on the chunk?

The NET FORCE on the chunk points along the y -axis, since the x -components cancel ($F_{\text{net},x} = F_{R,x} + F_{L,x} \approx +\tau - \tau = 0$). Therefore, we find:

$$F_{\text{net}} = F_{\text{net},y} = F_{R,y} + F_{L,y} = 6.7\text{N} - 60\text{N} \approx -53.3\text{N}$$

\Rightarrow The CLOSEST answer for the MAGNITUDE $|F_{\text{net}}|$ is (F) 53 N.

F

(b) Acceleration and Curvature (7 points) *Challenge Problem!*

Using your answer from part (a) (we won't penalize propagation of your errors):

- Find the acceleration ~~a_y~~ of the chunk. (Write out your numerical answer including units. This part is not multiple choice!)
 magnitude of the y-component of the

Using $\vec{F}_{net} = m_{ch} \vec{a}$ for the chunk, we find:

$$|a_y| = \frac{|F_{net,y}|}{m_{ch}} = \frac{|F_{net}|}{\mu \cdot \Delta} \approx \frac{53 \text{ N}}{(0.6 \text{ kg/m}) \times 2.5 \text{ m}}$$

$$\approx \boxed{35.3 \text{ m/s}^2}$$

- Using the wave equation, estimate the curvature $\left(\frac{\partial^2 y}{\partial x^2}\right)$ of the chunk. (Write out your numerical answer including units. This part is not multiple choice!)

From the wave equation for the chunk we find:

$$\tau \frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2} \Rightarrow \left| \frac{\partial^2 y}{\partial x^2} \right| = \frac{\mu}{\tau} |a_y|$$

$$\approx \frac{0.6 \text{ kg/m}}{1000 \text{ N}} \times 35.3 \text{ m/s}^2$$

$$\approx \boxed{0.021 \text{ m}^{-1}}$$

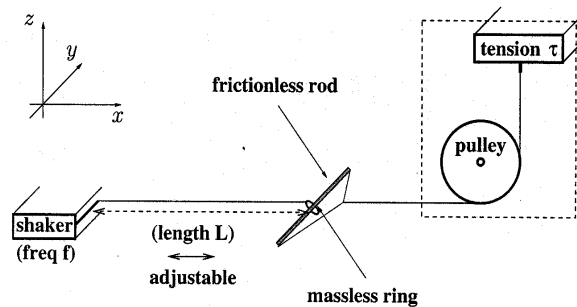


Figure 5: Lab Experiment I.

5 Problem 5: Lab Experiment I

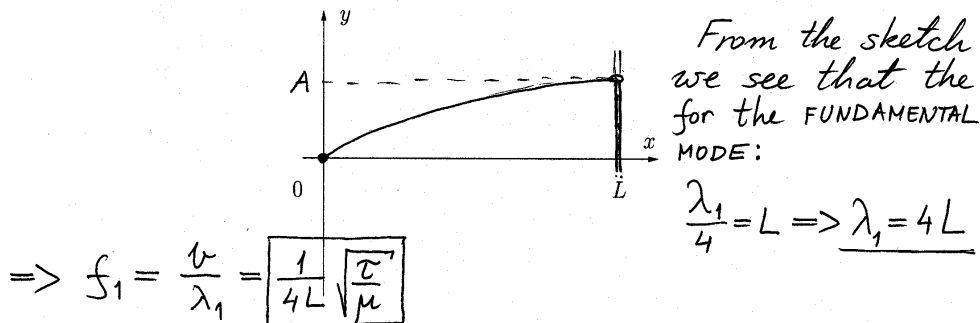
[21 points]

In your first laboratory experiment, you studied a vibrating string with two fixed ends. Consider now a modification of the experiment in which the right hand end of the string is attached to a small massless ring which slides up and down a horizontal (in and out of the page) post without friction. The post is attached through additional strings to a pulley on the other side of which spring scales (like the ones in lab) provide constant tension (see Figure 5).

As in lecture, L is the length of the string between the rod and the shaker, f is the frequency of the shaker, τ is the string tension and μ is the mass per unit length of the string. Assume μ is constant throughout the problem. The shaker vibrates the string *in and out* of the page. Assume that the left hand end is *fixed* at the shaker while the horizontal motion (along the y -direction) of the right hand end has no constraint, i.e., the right hand end is *free*.

(a) The Fundamental Mode (7 points)

Sketch the shape of the standing wave corresponding to the *lowest (fundamental) mode*. Find an expression for the lowest (fundamental) frequency f_1 in terms of L , τ and μ .



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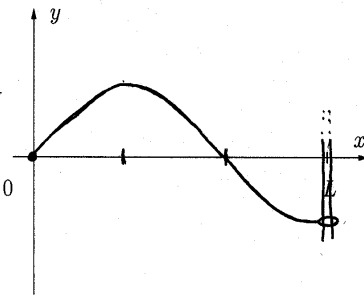
(b) Higher Modes at Fixed Length (7 points)

Sketch the shape of standing wave on the string corresponding to the *second mode*. What should the frequency f_2 of the shaker be for this standing wave mode to be formed? (Express f_2 in terms of L, τ and μ .)

From the sketch we see that for the *SECOND* mode:

$$\frac{3}{4}\lambda_2 = L$$

$$\Rightarrow \lambda_2 = \frac{4}{3}L$$



Thus:

$$f_2 = \frac{v}{\lambda_2} = \frac{1}{\left(\frac{4L}{3}\right)} \sqrt{\frac{\tau}{\mu}}$$

$$f_2 = \frac{3}{4L} \sqrt{\frac{\tau}{\mu}}$$

(c) Higher Modes at Fixed Frequency (7 points)

Now suppose that you can vary the length of the the string. Find the string length ℓ at which the 2nd standing wave mode will be formed, if the frequency of the shaker is fixed at f_1 and the tension τ remains constant. (Express ℓ in terms of ~~the~~ original length L .)

Since we are interested in the *SECOND* mode again, we can use directly the sketch of part (b):

$\lambda_2 = \frac{4}{3}\ell$, where ℓ is the new length of the string. Demanding that $f = \frac{v}{\lambda_2}$ be equal to the frequency found in (a), we get:

$$\frac{1}{4L} \sqrt{\frac{\tau}{\mu}} = f = \frac{3}{4\ell} \sqrt{\frac{\tau}{\mu}} \Rightarrow \ell = 3L$$

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6 Problem 6: Sound Waves versus Waves on a String [7 points]

The following equation is true for strings:

$$-\frac{\partial}{\partial x} \left[\tau \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \right] + \frac{\partial}{\partial t} \left[\frac{1}{2} \tau \left(\frac{\partial y}{\partial x} \right)^2 + \frac{1}{2} \mu \left(\frac{\partial y}{\partial t} \right)^2 \right] = 0. \quad (6.1)$$

It expresses conservation of energy for waves propagating on a string.

What equation expresses conservation of energy for sound waves?

Hint: Use the unified picture for string and sound waves presented in class and take a good guess!

The D. of F. describing waves on a string is $y(x,t)$, while the corresponding D. of F. for SOUND WAVES is the sound displacement function $S(x,t)$. From comparing the wave equations for string and sound (see the Formula Sheet for instance), we find that the mathematical analogue of τ in the case of sound is the BULK MODULUS B and the analogue of μ is the VOLUME MASS DENSITY ρ . Thus, conservation of energy for SOUND WAVES is expressed by the equation:

$$-\frac{\partial}{\partial x} \left[B \frac{\partial S}{\partial x} \frac{\partial S}{\partial t} \right] + \frac{\partial}{\partial t} \left[\frac{1}{2} B \left(\frac{\partial S}{\partial x} \right)^2 + \frac{1}{2} \rho \left(\frac{\partial S}{\partial t} \right)^2 \right] = 0$$

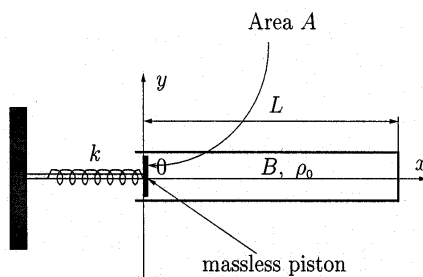


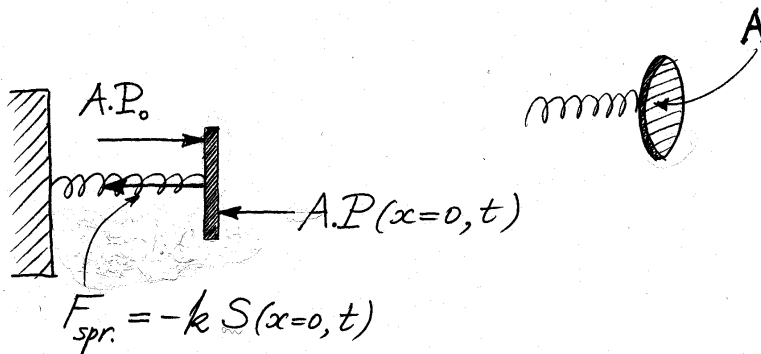
Figure 6: Generalized boundary conditions for a sound tube.

7 Problem 7: Generalized Boundary Conditions for a Sound Tube [14 points]

A tube of length L is filled with air of bulk modulus B and mass density ρ_0 at atmospheric pressure. One end of the tube ($x = L$) is closed while the other end ($x = 0$) is attached to a massless piston of area A that can slide freely (without friction) along the tube. (We consider low amplitude waves so that you can take $L \approx \text{const.}$) The piston is attached to an ideal spring of spring constant k that exerts a horizontal force on the piston; the spring is relaxed when the piston is at $x = 0$. (See Figure 6.) We denote the sound displacement inside the tube as $S(x, t)$ and the pressure inside the tube as $P(x, t)$.

(a) Air Pressure and Displacement of the Piston (7 points)

Draw a free body diagram for the piston, indicating the directions and the magnitudes of all the forces, using no quantities other than $P(x = 0, t)$, $S(x = 0, t)$, k , A , ρ_0 , L , and P_0 .



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(b) Boundary Condition at the Piston (7 points) *Challenge Problem!*

Write the *Equation of Motion for the piston* in terms of the *degrees of freedom*, i.e., the displacement function $S(x, t)$ and its derivatives evaluated at $x = 0$, using no quantities other than $P(x = 0, t)$, $S(x = 0, t)$, k , L , A , B , ρ_0 , and P_0 .

Let us write $\vec{F}_{net} = m_p \vec{a}_p$ for the piston using our free body diagram from part (a):

$$F_{net, x} = -kS(x=0, t) + A \cdot P_0 - A \cdot P(x=0, t) = m_p a_{p, x}$$

Since $m_p \rightarrow 0$ (massless piston), we have:

$$-kS(x=0, t) + AP_0 - AP(x=0, t) = 0$$

Expressing everything in terms of the D. of F. $S(x, t)$ and its derivatives evaluated at $x=0$, through $P(x, t) = P_0 - B \frac{\partial S}{\partial x}(x, t)$, we obtain:

$$-kS(x=0, t) + AP_0 - AP_0 + AB \frac{\partial S}{\partial x}(x=0, t) = 0$$

$$\Rightarrow \boxed{-kS(x=0, t) + A \cdot B \cdot \frac{\partial S}{\partial x}(x=0, t) = 0}$$

is the generalized boundary condition for the sound tube at $x=0$.