**Physics 214** **Fall 2001**

# Problem Set # 10

**Reading on current material and for coming lectures:** *Young & Freedman*, Sections 41.5–41.6, 42.1–42.7, 43.1–43.2.

## Problems: (due 12/11)

*Skills to be mastered:*

- *Be able to use your knowledge about diffraction of light to analyze diffraction of electrons;*
- *Be able to determine energies and the wavefunctions of bound states of a quantum particle for simple one-dimensional potentials;*
- *Be able to solve quantum scattering problems in one dimension;*
- *Be able to compute probability densities and probability density fluxes for quantum particles;*
- *Understand and be able to apply Heisenberg uncertainty principle in different situations.*

**1. Problem 1 (Electron Diffraction):**
In 1927 *Davisson and Germer* observed electron diffraction by accelerating electrons in a vacuum tube, similar to the one demonstrated in class, and then aiming them at a nickel (Ni) target. The electrons scattered from the atomic sites on the nickel surface diffract in a similar way to the light beams scattered off the $1/64''$ marks of the ruler used in Experiment 1 of Lab 3. In their experiment Davisson and Germer used voltage $V = 54$ V and the inter-atomic spacing of the Ni target was $d = 2.15$ Å.

If one is to repeat Davisson-Germer's experiment with a setup like the one discussed in lecture, in which the beam is *perpendicular* to the surface of the target,

(a) ***How many*** diffraction ***minima*** would one observe***?*** ***At what angles?***

(b) ***How many*** diffraction ***maxima*** would one observe***?***

**2. Problem 2 (Infinite Square Well Potential):**
A particle of mass $m$ is confined to a inpenetrable box of length $2a$. The box can be modeled by a **square well potential** of **infinite** depth, as the one shown on Figure 1.

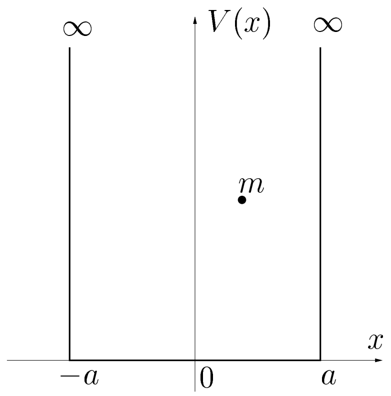


Figure 1: Infinite square well potential.

(a) ***Draw*** the ***wavefunctions*** of the ***three lowest-lying energy eigenstates***.

(b) ***Find*** the ***energies*** of the ***three lowest-lying energy eigenstates*** of the particle.

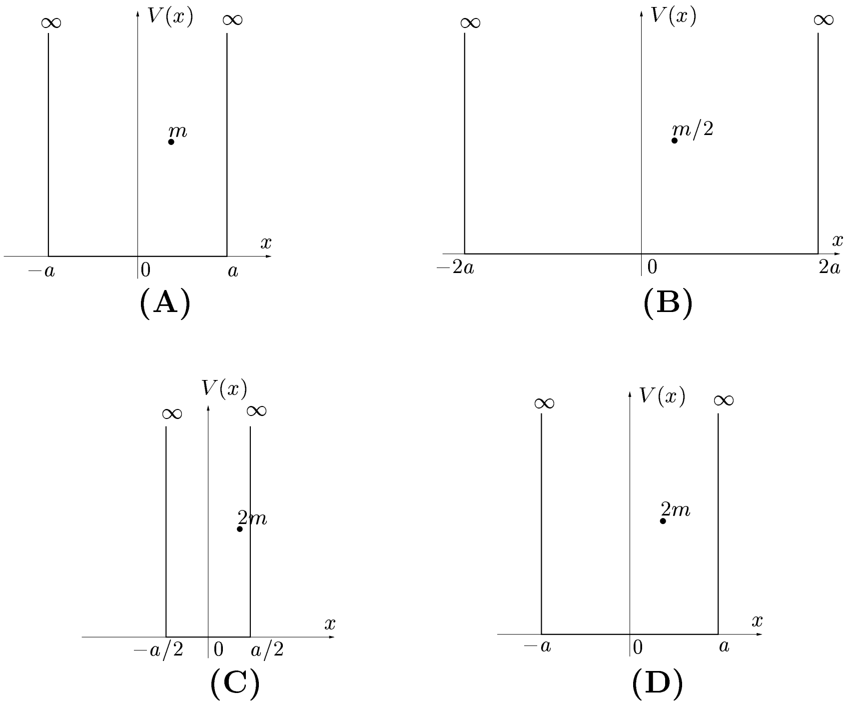


Figure 2: Square wells of different width versus particles of different masses.

(c) ***Which two*** of the configurations shown on Figure 2 have ***the same ground state energies?*** (**"ground state"** = the state of the lowest energy in a given system)

**3. Problem 3 (Step Potential):**
A beam of particles, each of mass $m$ and energy $E$ travelling to the left, encounters a **step potential** of height $V_0$ (see Figure 3).

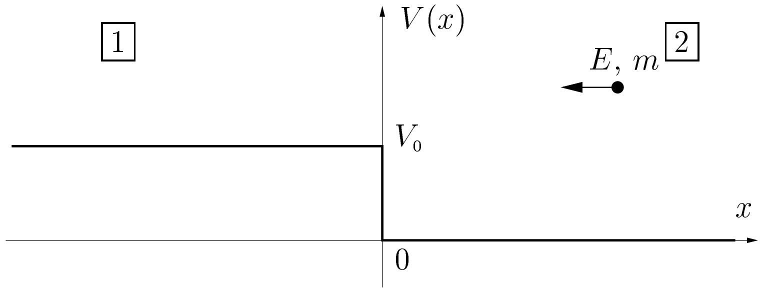


Figure 3: Traveling particle and step potential.

Suppose that the particles in the beam can be described by complex **plane waves**, so that the *incident* particle wave is $e^{-ik_2x}$, the *reflected* wave is $re^{ik_2x}$ and the *transmitted* wave is $te^{-ik_1x}$, where $r$ and $t$ are **reflection** and **transmission** amplitudes.

The particle wavefunction in the regions $x > 0$ and $x < 0$ has the forms:

$$\text{Region 2 } (x > 0): \qquad \psi_2(x) = \sqrt{n}\,\left[e^{-ik_2x} + re^{ik_2x}\right], \tag{1}$$

$$\text{Region 1 } (x < 0): \qquad \psi_1(x) = \sqrt{n}\,te^{-ik_1x}, \tag{2}$$

where the constant $n$ can be interpreted as the particle (or probability) density of the incident beam. (You can think of $\psi(x)$ to be the spatial part of the full (time-dependent) wavefunction: $\Psi(x,t) = \psi(x)e^{i\omega t}$.)

(a) ***Show that*** the wavevectors $k_1$ and $k_2$ in the two regions ***are given by*** $k_2 = \sqrt{2mE}/\hbar$ and $k_1 = \sqrt{2m(E - V_0)}/\hbar$. You may use De Broglie's relations: $E = \hbar\omega$, $p = \hbar k$, and the relationship between kinetic, potential, and total energy: $E = V(x) + p^2/2m$.

(b) Using your results of part (b) and the boundary conditions

$$\psi_2(0^+) = \psi_1(0^-) , \qquad \frac{d\psi_2}{dx}(0^+) = \frac{d\psi_1}{dx}(0^-) , \tag{3}$$

***compute*** $r$ ***and*** $t$ ***in terms of*** $E$, $m$, ***and*** $V_0$. ***Write down explicitly*** the particle wavefunction $\psi_1(x)$ and $\psi_2(x)$ in the two regions.

Hint: Clearly, eq. (2) represents the fact that the wavefunction in the region $x > 0$ is a superposition of an incident and a reflected wave, while the wavefunction in the region $x < 0$ is a pure transmitted wave. The problem is thus very similar to *Pr.3* of *PS#8*.

(c) Consider the cases $E > V_0$ and $E < V_0$. ***Describe what happens to your answers for*** $k_1$, $k_2$, $r$, ***and*** $t$ ***in each case. How is the wavefunction*** $\psi_1(x)$ ***in the*** $x < 0$ ***region different in the two cases?***

(d) Now imagine that the moving particle encounters a **step down** in the potential energy, such that $E > V_0$ (see Figure 4).

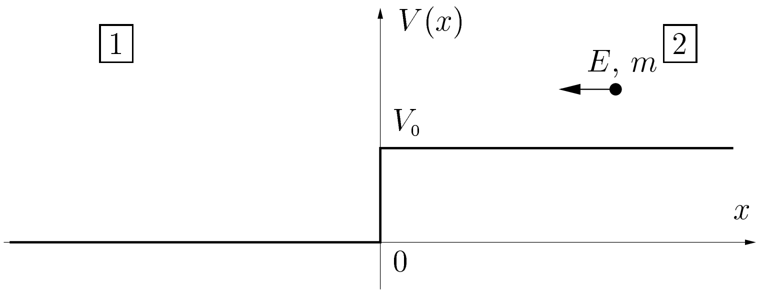


Figure 4: Traveling particle and step potential.

***What are the values,*** $r'$ ***and*** $t'$, ***of the reflection and transmission amplitudes in this case?***

Hint: You needn't do any new calculations here. Simply exchange $k_1$ and $k_2$ in you previous answers.

(e) The **probability density** for detecting a particle at position $x$ is defined as $\mathcal{P}(x) = |\psi(x)|^2$ and the **probability density flux** *for a pure traveling wave* as $\mathcal{F}(x) = \mp v|\psi(x)|^2$. (In the latter, $v$ is the classical speed of the particles, '$-$' corresponds to a left-moving wave, and '$+$' to a right-moving wave.) ***Show that***

$$|\mathcal{F}_i| = |\mathcal{F}_r| + |\mathcal{F}_t| , \tag{4}$$

i.e., the probability density flux of the incident wave is equal to the sum of the fluxes of the reflected and transmitted waves, and thus ***probability is conserved.***

Note: $\mathcal{P}(x)$ and $\mathcal{F}(x)$ are the mathematical analogues to the energy density $e(x)$ and power $P(x)$ in classical waves, even though their physical interpretation in quantum mechanics is completely different. This part therefore should be very similar mathematically to *Pr.3(b)* of *PS#9*, where we found that $|P_i| = |P_r| + |P_t|$ for the incident, reflected, and transmitted power.

(f) ***Write down*** $\mathcal{F}_t$ ***for all the cases considered in parts*** (c) ***and*** (d).

**4. Problem 4 (Scattering from a potential bump; tunneling):**
Now suppose that the particles beam of the previous problem encounters a **potential bump** of height $V_0$ and width $d$ (see Figure 5).

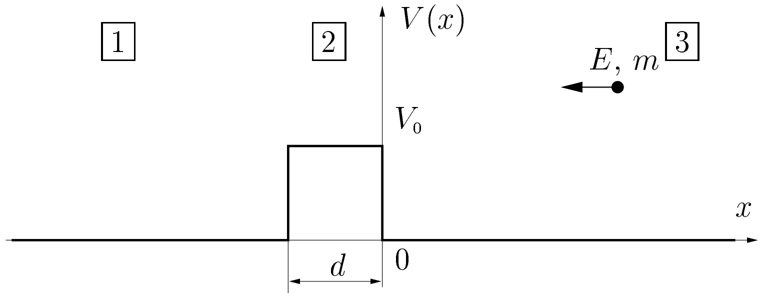


Figure 5: Traveling particle and step potential.

Use techniques similar to the ones in *Pr.4* of *PS#8* to find the reflection and transmission amplitudes, and thus to analyze the particle wavefunction in the regions $x < -d$ and $x > 0$:

(a) ***Show that the particle wavefunction in the regions*** $x > 0$ ***and*** $x < -d$ ***has the same general form as in eq.*** (2)***:***

$$\text{Region 3 } (x > 0): \qquad \psi_3(x) = \sqrt{n}\ \left[e^{-ik_3x} + r(d)\, e^{ik_3x}\right] \ , \tag{5}$$

$$\text{Region 1 } (x < -d): \qquad \psi_1(x) = \sqrt{n}\, t(d)\, e^{-ik_1x} \ , \qquad (k_1 = k_3 = k) \tag{6}$$

where $r(d)$ and $t(d)$ are the *'renormalized'* reflection and transmission coefficients. Using an infinite summation over reflected and transmitted plane waves, ***compute*** $r(d)$ ***and*** $t(d)$ ***in terms of*** $E$***,*** $m$***,*** $V_0$***, and*** $d$***.***

Hint: The reflection and transmission of a *single* plane wave through the boundary at $x = 0$ will be described by the $r$ and $t$ found in 3(c), and through the boundary at $x = -d$, by the $r'$ and $t'$ found in 3(d).

(b) ***Compute the probability density flux*** $\mathcal{F}_t$ ***of the wave transmitted into the*** $x < -d$ ***region, in the case*** $E < V_0$***.*** If $E < V_0$, a *classical* particle will bounce back from the potential wall at $x = 0$ and *would never make it* into the $x < -d$ region. ***Will a quantum particle make it there?***

Hint: First, note that the wavevector $k_2$ in Region 2 is *imaginary* when $E < V_0$, then keeping this fact in mind, compute $r(d)$ and $t(d)$, and at last, calculate the flux $\mathcal{F}_t$.

**5. Problem 5 (Heisenberg uncertainty principle and the ground state energy of the Hydrogen atom):**
In this problem we will use the *Heisenberg uncertainty principle* and simple considerations involving *energy* and *momentum* to estimate the ground state energy of the Hydrogen atom.

(a) Using the Heisenberg uncertainty principle, ***argue that if the separation of the electron and the nucleus in the Hydrogen atom is of order*** $r$***, then the momentum of the electron is of order*** $\dfrac{\hbar}{r}$***.***

(b) ***Write down the total energy of the electron as a function of*** $r$***.*** Use your result of part (a) to find the *kinetic energy* of the electron, and the Coulomb formula $V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$ for its *potential energy.*

(c) ***Find the separation*** $r_0$ ***at which the energy*** $E(r)$ ***has a minimum. What is the value*** $E_{\text{min}} = E(r_0)$***?*** How do your answers compare with the *Bohr radius,* $r_B$, and the ground state energy $E_1$ of Hydrogen, given in *Young & Freedman, eqs. (43-3) and (43-8)*?