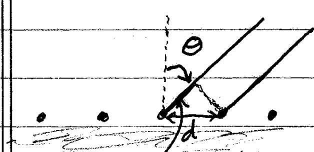
P214 HOMEWORK SET #10 SOLUTIONS

TA: MVA

PROBLEM 1:



$\Delta r = d\sin\theta$

THE SETUP IS DEPICTED ABOVE. A BEAM OF ELECTRONS IS INCIDENT ON A REGULARLY SPACED ARRAY OF POINT-LIKE SCATTERERS, WHICH ACT AS A DIFFRACTION GRATING.

(PART B) THE CONDITION FOR CONSTRUCTIVE INTERFERENCE IS:

$k\Delta r = n2\pi \Rightarrow d\sin\theta = n\lambda$

USING THIS FORMULA, WITH $\lambda = \frac{2\pi}{k}$:

THE MOMENTUM OF THE ELECTRONS IS $p = \hbar k$.

THE ELECTRONS ARE ACCELERATED THROUGH A POTENTIAL V, AND ACQUIRE KINETIC ENERGY eV (e - ELECTRON CHARGE, V - POTENTIAL). THEN:

$eV = \frac{p^2}{2m} \Rightarrow p = \sqrt{2meV} \Rightarrow k = \frac{\sqrt{2meV}}{\hbar} \Rightarrow \lambda = \frac{2\pi\hbar}{\sqrt{2meV}} =$

$= 1.67\times10^{-10}\,m$. THEN:

$\sin\theta = n\frac{\lambda}{d} = (0.777)n$

FOR A PERPENDICULAR INCIDENT BEAM, WE LOOK FOR MAXIMA AT $n = 1, 2, 3, \dots$

$\sin\theta_1 = .777 \Rightarrow \theta_1 = 51^\circ$

$\sin\theta_2 = 1.55 \Rightarrow$ OBSERVE NO HIGHER ORDER MAXIMA

($\sin\theta_2 = 1.55$ HAS NO REAL SOLUTION IN θ)

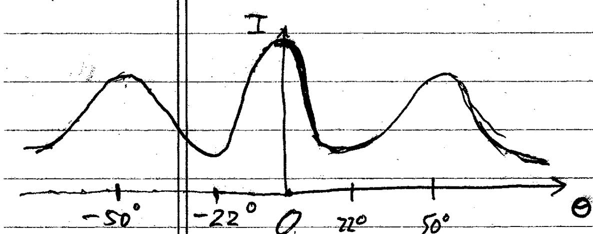
(PART A) THE CONDITION FOR MINIMA (AS ABOVE) IS:

$d\sin\theta = n\frac{\lambda}{2}, \quad n = 1, 3, 5, \ldots$

THEN: $\sin\theta_1 = 0.38 \Rightarrow \theta_1 = 22^\circ$

$\sin\theta_3 = 1.14 \Rightarrow$ OBSERVE NO HIGHER ORDER MINIMA.

THUS, THE PLOT OF OBSERVED INTENSITY VERSUS ANGLE IS:



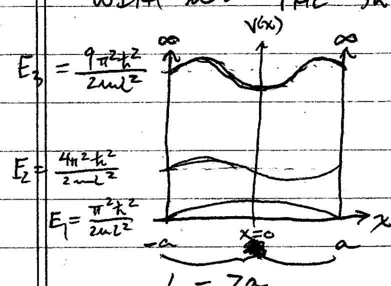
PROBLEM 2:

IT WAS SHOWN IN LECTURE THAT THE WAVEFUNCTIONS FOR THE PARTICLE IN A BOX ARE THE STANDING WAVE MODES FOR WAVES ON A STRING FIXED AT BOTH ENDS. ALTERNATIVELY, THIS RESULT CAN BE DERIVED USING THE SCHRÖDINGER EQUATION. EITHER WAY:

$$\lambda_n = \frac{2L}{n}$$

ALSO FROM THE SCHRÖDINGER EQ., OR BY USING DE BROGLIE: $k = \frac{2\pi}{\lambda} = \frac{n\pi}{L}$; $p = \hbar k$, and $E = \frac{p^2}{2m}$ INSIDE THE BOX. SUBSTITUTING IN GIVES: $E = \frac{n^2\pi^2\hbar^2}{2mL^2}$. THUS, THE ENERGIES RISE QUADRATICALLY WITH n. THE SKETCH LOOKS LIKE:

(PARTS A & B)



$\boxed{\text{(IN ALL THE WORK ABOVE, } L = 2a\text{)}}$

(PART C)

PART C CONSIDERS FOUR "BOXES":

A — $m' = m$, $L' = 2a$ $\Rightarrow$ $E_{1,A} = \frac{\pi^2\hbar^2}{2m\times 4a^2} = \frac{\pi^2\hbar^2}{8ma^2}$

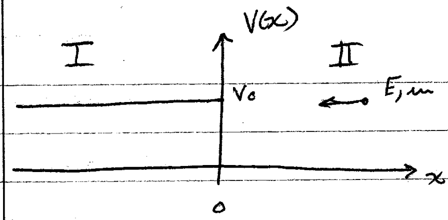
B — $m' = \frac{1}{2}m$, $L' = 4a$ $\quad E_{1,B} = \frac{\pi^2\hbar^2}{2\cdot\frac{1}{2}m\cdot 16a^2} = \frac{\pi^2\hbar^2}{16ma^2}$

C — $m' = 2m$, $L' = a$ $\quad E_{1,C} = \frac{\pi^2\hbar^2}{2\cdot 2m a^2} = \frac{\pi^2\hbar^2}{4ma^2}$

D — $m' = 2m$, $L' = 2a$ $\quad E_{1,D} = \frac{\pi^2\hbar^2}{2\times 2m\times 4a^2} = \frac{\pi^2\hbar^2}{16ma^2}$

THUS $\underline{E_{1,B} = E_{1,D}}$.

PROBLEM 3:



THE PROBLEM GIVES THE FORM OF THE WAVEFUNCTIONS IN REGIONS I & II:

$$\text{II} — \chi_{II}(x) = \sqrt{n}\left(e^{-ik_2x} + r e^{ik_2x}\right)$$
$$\text{I} — \chi_{I}(x) = \sqrt{n}\, t e^{-ik_1x}$$

(PART A) THE WAVEVECTORS IN THE TWO REGIONS CAN BE DETERMINED USING DE BROGLIE OR THE SCHRÖDINGER EQ. USING DE BROGLIE:

IN REGION II, $E = \frac{p^2}{2m}$. $p = \hbar k$, so $\frac{\hbar^2 k^2}{2m} = E \Rightarrow$ $k_{II} = \frac{\sqrt{2mE}}{\hbar}$.

IN REGION I, $E = \frac{p^2}{2m} + V_0 \Rightarrow \frac{\hbar^2 k^2}{2m} = E - V_0 \Rightarrow$ $k_{I} = \frac{\sqrt{2m(E-V_0)}}{\hbar}$.

(PART B) THE BOUNDARY CONDITIONS TO BE USED ARE ALSO GIVEN:
B.C. #1: $\chi_{II}(0^+) = \chi_{I}(0^-)$, WHERE + & − DENOTE ONE-SIDED LIMITS, SINCE χ_{II} & χ_{I} ARE ONLY DEFINED ON THE RIGHT & LEFT-HAND SIDES OF THE BOUNDARY, RESPECTIVELY.
B.C. #2: $\frac{d\chi_{II}}{dx}(0^+) = \frac{d\chi_{I}}{dx}(0^-)$
SUBSTITUTING χ_I & χ_{II} INTO B.Cs #1 & 2:

$$\chi_{II}(0) = \chi_I(0) \Rightarrow \sqrt{u}\,(1+r) = \sqrt{u}\,t \Rightarrow$$
$$t = 1+r$$

$$\frac{d\chi_{II}}{dx} = \sqrt{u}\left(-ik_2 e^{-ik_2 x} + rik_2 e^{ik_2 x}\right)$$

$$\frac{d\chi_I}{dx} = \sqrt{u}\,t(-ik_1)\,e^{-ik_1 x} \Rightarrow$$

$$\frac{d\chi_{II}}{dx}(0) = \frac{d\chi_I}{dx}(0) \Rightarrow \sqrt{u}\left(-ik_2 + rik_2\right) = \sqrt{u}\,t\left(-ik_1\right)$$

$$\Rightarrow \quad -k_2 + rk_2 = -tk_1$$

So: $\quad 1+r = t$

AND $\quad (1-r)k_2 = +tk_1 \Rightarrow (1-r)k_2 = (1+r)k_1 \Rightarrow$

$$r(k_1+k_2) = k_2 - k_1 \Rightarrow r = \frac{k_2-k_1}{k_1+k_2}$$

THEN $\quad t = \underbrace{\frac{k_1+k_2}{k_1+k_2}}_{(1)} + \frac{k_2-k_1}{k_1+k_2} = \frac{2k_2}{k_1+k_2}$

(PART C) SUBSTITUTING FOR k_1 & k_2:

$$\chi_{II} = \sqrt{u}\left(e^{-i\frac{\sqrt{2mE}}{\hbar}x} + \overset{r}{\frac{\sqrt{2mE}-\sqrt{2m(E-V_0)}}{\sqrt{2mE}+\sqrt{2m(E-V_0)}}}\, e^{i\frac{\sqrt{2mE}}{\hbar}x}\right)$$

$$\chi_I = \sqrt{u}\,\frac{2\sqrt{2mE}}{\sqrt{2m(E-V_0)}+\sqrt{2mE}}\, e^{-i\frac{\sqrt{2m(E-V_0)}}{\hbar}x}$$

FOR $E < V_0$ K_1 BECOMES IMAGINARY, SINCE $\sqrt{2m(E-V_0)} \to \sqrt{-2m|E-V_0|} = i\sqrt{2m|E-V_0|}$. K_2 REMAINS REAL, BUT r, & t BECOME COMPLEX. WITH K_1 IMAGINARY, χ_I BECOMES REAL — THE SOLUTION CEASES TO OSCILLATE IN REGION I! IN THIS REGION, IT TAKES THE FORM OF A DECAYING EXPONENTIAL:

$$\chi_I = \sqrt{u}\,t\,e^{-i^2\frac{\sqrt{2m|E-V_0|}}{\hbar}x} = \sqrt{u}\,t\,e^{\frac{\sqrt{2m|E-V_0|}}{\hbar}x} = \sqrt{u}\,t\,e^{-\frac{\sqrt{2m|E-V_0|}}{\hbar}|x|} \quad \text{(SINCE } x<0 \text{ IN REGION I)}$$

FOR $E > V_0$, $k_1, k_2, r, \text{ \& } t$ ARE ALL REAL. THUS, THE WAVEFUNCTIONS OSCILLATE IN BOTH REGIONS.

(PART D) AS GIVEN IN THE HINT IN THE PROBLEM, SINCE THE REGION WITH THE POTENTIAL HAS BEEN SWITCHED, THE ROLES OF k_1 & k_2 IN r & t MERELY NEED TO BE EXCHANGED. THEN:

$$r = \frac{k_2 - k_1}{k_1 + k_2} \Rightarrow r' \to \frac{k_1 - k_2}{k_2 + k_1}, \text{ AND:}$$

$$t = \frac{2k_2}{k_1 + k_2} \Rightarrow t' \to \frac{2k_1}{k_2 + k_1}$$

(PART E) THE PROBABILITY DENSITY "FLUX" IS DEFINED TO BE:

$$F(x) = \underset{\text{RIGHT-MOVING}}{\overset{\text{LEFT-MOVING}}{\mp}} v|\chi|^2,$$ WHERE v IS THE "CLASSICAL" VELOCITY.

THE "CLASSICAL" VELOCITY MEANS THE VELOCITY THE PARTICLE MOVES WITH IF IT WERE TREATED IGNORING ITS WAVE-PROPERTIES. THEN $v = \frac{\overset{\text{MOMENTUM}}{p}}{\underset{\text{MASS}}{m}}$, BY DEFINITION. BUT WE CAN RELATE THE MOMENTUM TO THE WAVE-PROPERTIES OF THE PARTICLE, WHICH WILL ALLOW US TO PROVE THE REQUIRED RESULT.

$$F_i = -v_i|\chi_i|^2 = -\frac{p_i}{m}|e^{-ik_2 x}|^2 = -\frac{\hbar k_2}{m}$$

$$F_r = +\frac{p_r}{m}|\chi_r|^2 = \frac{\hbar k_2}{m}|re^{ik_2x}|^2 = \frac{\hbar k_2}{m}|r|^2 =$$

$$= \frac{\hbar k_2}{m}\frac{|k_2-k_1|^2}{|k_2+k_1|^2} = \left(\frac{k_2^2+|k_1|^2-2\,\mathrm{Re}\,k_1k_2}{|k_1+k_2|^2}\right)\frac{\hbar k_2}{m}$$

$$F_t = -\frac{\hbar\,\mathrm{Re}\,k_1}{m}\frac{4k_2^2}{|k_1+k_2|^2}$$

The reason FOR THE RELATION $F_t = \frac{\hbar\,\mathrm{Re}\,k_1}{m}$ IS THAT FOR IMAGINARY k_1, WE NO LONGER HAVE A PLANE WAVE TRAVELING THAT CAN CONTRIBUTE TO THE PROBABILITY FLUX. IN THIS CASE, $F_t = 0$. SINCE $\mathrm{Re}\,k_1 = 0$ FOR k_1 IMAGINARY, THIS GIVES THE RIGHT RESULT IN THIS CASE. THEN:

$$|F_r| + |F_t| = \left(\frac{k_2^2+|k_1|^2-2\,\mathrm{Re}\,k_1k_2}{|k_1+k_2|^2}\right)\frac{\hbar k_2}{m} + \frac{\hbar\,\mathrm{Re}\,k_1\,4k_2^2}{|k_1+k_2|^2} =$$

$$= \frac{\hbar k_2}{m}\left(\frac{|k_1|^2+k_2^2+2\,\mathrm{Re}\,k_1k_2}{|k_1+k_2|^2}\right) = \frac{\hbar k_2}{m}\frac{|k_1+k_2|^2}{|k_1+k_2|^2} = \frac{\hbar k_2}{m} =$$

$= |F_i|$, AS REQUIRED.

(PART F) FOR PART C: $t = \frac{4k_2}{k_1+k_2}$, $r = \frac{k_2-k_1}{k_1+k_2}$

CASE i) $E > V_0$

$$F_t = -\frac{\hbar k_1}{m}\frac{4k_2^2}{k_2^2+|k_1|^2+2k_1k_2}\ ;\ k_1 = \frac{\sqrt{2m(E-V_0)}}{\hbar},\ k_2 = \sqrt{2mE}/\hbar$$

CASE ii) AS DISCUSSED ABOVE, $F_t = 0$. $(E < V_0)$

FOR PART D:

CASE i) $E > V_0$ $\quad F_t = -\frac{\hbar k_2}{m}|t'|^2 = -\frac{\hbar k_2}{m}\frac{4(k_1)^2}{k_1^2+k_2^2+2k_1k_2}$, where

$$k_1 = \sqrt{2m(E-V_0)}/\hbar,\ k_2 = \sqrt{2mE}/\hbar$$

CASE ii) $E < V_0$ $\quad F_t = -\frac{\hbar k_2}{m}\frac{4|k_1|^2}{k_2^2+|k_1|^2}$, where

$$k_1 = i\sqrt{2m|E-V_0|}/\hbar,\ k_2 = \sqrt{2mE}/\hbar$$

④ (a) • $\psi_i(x) = \sqrt{n}\, e^{-ik_3 x}$;

• $\psi_r(x) = \sqrt{n}\Big[r e^{ik_3 x} + t e^{ik_2 d} r' e^{ik_2 d} t' e^{ik_3 x}$
$\qquad + t e^{ik_2 d} r' e^{ik_2 d} \left(r' e^{ik_2 d} r' e^{ik_2 d}\right) t' e^{ik_3 x} + \dots \Big]$

$$= \sqrt{n}\, e^{ik_3 x}\left[r + r' t t' e^{2ik_2 d}\left(1 + r'^2 e^{2ik_2 d} + \left(r'^2 e^{2ik_2 d}\right)^2 + \dots\right)\right]$$

$$= \sqrt{n}\, e^{ik_3 x}\left[r + r' \frac{t t' e^{2ik_2 d}}{1 - r'^2 e^{2ik_2 d}}\right].$$

From Problem 3, $r' = -r$ and $tt' = 1 - r^2$, so:

<u>In Region 3:</u> $\psi_3(x) = \sqrt{n}\left[e^{-ik_3 x} + r(d) e^{ik_3 x}\right]$,

where $\boxed{r(d) = r\, \dfrac{1 - e^{2ik_2 d}}{1 - r^2 e^{2ik_2 d}}}$ with:

$$r = \frac{k_2 - k_1}{k_2 + k_1}, \quad k_1 = k_3 = \frac{\sqrt{2mE}}{\hbar}, \quad k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

• $\psi_t(x) = \sqrt{n}\, e^{-ik_1 x}\Big[t e^{ik_2 d} t' + t e^{ik_2 d} r' e^{ik_2 d} r' e^{ik_2 d} t'$
$\qquad + t e^{ik_2 d}\left(r'^2 e^{2ik_2 d}\right)^2 t' + \dots \Big]$

$$= \sqrt{n}\, e^{-ik_1 x}\, t e^{ik_2 d} t' \left[1 + \left(r'^2 e^{2ik_2 d}\right) + \left(r'^2 e^{2ik_2 d}\right)^2 + \dots\right]$$

$$= \sqrt{n}\, e^{-ik_1 x} \frac{t t' e^{ik_2 d}}{1 - r^2 e^{2ik_2 d}}, \text{ so:}$$

In Region 1: $\psi_1(x) = \sqrt{n}\, t(d) e^{-ik_1 x}$,

where $\boxed{t(d) = \frac{tt' e^{ik_2 d}}{1 - r^2 e^{2ik_2 d}}}$ with:

$$r = \frac{k_2 - k_1}{k_2 + k_1}, \quad t = \frac{2k_1}{k_1 + k_2}, \quad t' = \frac{2k_2}{k_1 + k_2},$$

$$k_1 = k_3 = \frac{\sqrt{2mE}}{\hbar}, \quad k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

Note that the "$ik_2 d$" factors have different sign compared to Problem 4, PS #8, because there we had the opposite conventions for right-moving and left-moving waves.

(b) In the case $E < V_0$, the wavevector k_2 becomes IMAGINARY:

$$\underline{\underline{k_2}} = \frac{\sqrt{2m(E - V_0)}}{\hbar} = i \frac{\sqrt{2m(V_0 - E)}}{\hbar} = \underline{\underline{i\alpha_2}} \quad (\alpha_2 \text{ is REAL})$$

and t, t' and r become COMPLEX, so we need to be careful with the complex conjugation while computing S_t.

Let's see how far we can get without using the explicit form of t, t' and r:

$$\mathfrak{S}_t = -v|\psi_1(x)|^2 = -nv|t(d)e^{-ik_1x}|^2$$

$$= -nv|t(d)|^2 \overbrace{|e^{-ik_1x}|^2}^{=1} = -nv\,t^*(d)\,t(d)$$

$$= -nv\left(\frac{tt'e^{-\alpha_2 d}}{1-r^2e^{-2\alpha_2 d}}\right)^*\left(\frac{tt'e^{-\alpha_2 d}}{1-r^2e^{-2\alpha_2 d}}\right)$$

$$= \frac{-nv|tt'|^2e^{-2\alpha_2 d}}{\left(1-r^2e^{-2\alpha_2 d}\right)\left(1-(r^2)^*e^{-2\alpha_2 d}\right)}$$

$$= \frac{-nv|tt'|^2e^{-2\alpha_2 d}}{1+|r^2|^2e^{-4\alpha_2 d}-\underbrace{\left(r^2+(r^2)^*\right)}_{2Re[r^2]}e^{-2\alpha_2 d}}$$

$$\Rightarrow \boxed{\mathfrak{S}_t = \frac{-nv|tt'|^2e^{-2\alpha_2 d}}{1+|r^2|^2e^{-4\alpha_2 d}-2Re[r^2]e^{-2\alpha_2 d}}}$$

Finally, let's substitute the expressions for t, t', and r:

$$t = \frac{2k_1}{k_1+k_2} = \frac{2}{1+i(\alpha_2/k_1)}; \quad t' = \frac{2k_2}{k_1+k_2} = \frac{2}{1-i(k_1/\alpha_2)}$$

$$r = \frac{k_2-k_1}{k_2+k_1} = \frac{i\alpha_2-k_1}{i\alpha_2+k_1}$$

We then get:

$$|tt'|^2 = \left|\frac{4}{2+i\left(\frac{\alpha_2}{k_1}-\frac{k_1}{\alpha_2}\right)}\right|^2 = \underline{\frac{16}{4+\varkappa^2}} \; ; \; \underline{\varkappa \equiv \frac{\alpha_2}{k_1}-\frac{k_1}{\alpha_2}}$$

$$|r^2|^2 = (r^2)(r^2)^* = \left(\cancel{\frac{i\alpha_2-k_1}{i\alpha_2+k_1}}\right)^2 \cdot \left(\cancel{\frac{-i\alpha_2-k_1}{-i\alpha_2+k_1}}\right)^2 = \underline{1}$$

$$Re[r^2] = Re[1-tt'] = 1-Re[tt'] = 1-Re\left[\frac{4}{2+i\varkappa}\right]$$

$$= 1-4\frac{Re(2-i\varkappa)}{4+\varkappa^2} = \underline{\frac{\varkappa^2-4}{\varkappa^2+4}} \; ; \; \underline{\varkappa \equiv \frac{\alpha_2}{k_1}-\frac{k_1}{\alpha_2}}$$

Substituting in the flux:

$$\mathcal{F}_t = -nv\frac{16}{\varkappa^2+4}\cdot\frac{e^{-2\alpha_2 d}}{1+e^{-4\alpha_2 d}-2\left(\frac{\varkappa^2-4}{\varkappa^2+4}\right)e^{-2\alpha_2 d}}$$

$$= -nv.\cancel{16}^{\,8}\frac{1}{(\varkappa^2+4)\underbrace{\left(e^{2\alpha_2 d}+e^{-2\alpha_2 d}\right)}_{\cancel{2}\cosh(\alpha_2 d)}-\cancel{2}(\varkappa^2-4)}$$

Finally: $$\boxed{\mathcal{F}_t = -nv\frac{8}{(\varkappa^2+4)\cosh(\alpha_2 d)-(\varkappa^2-4)}}$$

where $v = \frac{\hbar k_1}{m} = \sqrt{2E/m}$

$$\varkappa = \frac{\alpha_2}{k_1}-\frac{k_1}{\alpha_2} = \sqrt{\frac{V_0}{E}-1}-\frac{1}{\sqrt{\frac{V_0}{E}-1}}$$

We see that $\mathcal{F}_t$ is a NON-ZERO REAL NUMBER

$\Longrightarrow$ Some particles will "TUNNEL" into Region 1.

In the limit $d \to 0$, our expression for the flux reduces to:

$$\lim_{d \to 0} \mathfrak{F}_t = -nv \frac{8}{(\varkappa^2+4)\underbrace{\cosh(0)}_{1} - (\varkappa^2-4)}$$

$$= -nv \frac{8}{\cancel{\varkappa^2}+4-\cancel{\varkappa^2}+4} = -nv \;!$$

This is expected, since when $d=0$ there is no barrier and ALL the particles will be transmitted: $\underbrace{\mathfrak{F}_t = \mathfrak{F}_i = -nv}_{(\text{when } d=0)}$

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5) a) $\Delta x \Delta p \gtrsim 2\pi\hbar$

If the electron orbits at a radius of order r, the uncertainty in its position is of order $2\pi r$

$$2\pi r \Delta p \gtrsim 2\pi\hbar \qquad \Delta p \gtrsim \frac{\hbar}{r}$$

The electron's momentum is of order $\frac{\hbar}{r}$.

b) $KE = \frac{p^2}{2m} = \frac{\hbar^2}{2mr^2} \qquad V(r) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$

$$E = \frac{\hbar^2}{2mr^2} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

c) $\frac{dE}{dr} = -2\frac{\hbar^2}{2mr^3} + \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = 0$

$$-\frac{\hbar^2}{m} + \frac{1}{4\pi\epsilon_0}e^2 r = 0 \qquad \frac{e^2}{4\pi\epsilon_0} r = \frac{\hbar^2}{m}$$

$$\boxed{r = \frac{\hbar^2}{m}\frac{4\pi\epsilon_0}{e^2}}$$

$$E = \frac{\hbar^2}{2m}\left(\frac{m^2e^4}{\hbar^4(4\pi\epsilon_0)^2}\right) - \frac{e^2}{4\pi\epsilon_0}\left(\frac{me^2}{\hbar^2 4\pi\epsilon_0}\right) = \frac{me^4}{2\hbar^2(4\pi\epsilon_0)^2} - \frac{2me^4}{2\hbar^2(4\pi\epsilon_0)^2}$$

$$\boxed{E_{min} = -\frac{me^4}{2\hbar^2(4\pi\epsilon_0)^2}}$$

These two quantities match what is found in Young $ Freedman.