**Physics 214** **Fall 2001**

# Problem Set # 3

## Reading for coming lectures: *Young & Freedman*, Sections 20.1—20-4; 19.6—19.9; 20.5.

## Problems: (due 9/25)

*Skills to be mastered:*

- *be able to convert between wavelength ($\lambda$), wavenumber or spatial frequency ($\kappa = 1/\lambda$), and wave-vector ($k = 2\pi/\lambda$);*
- *given the wave speed ($v$) and any one of the quantities ($T$, $f$, $\omega$), be able to find any one of ($\lambda$, $\kappa$, $k$), and vice versa;*
- *be able to distinguish between speed of wave propagation and particle speed.*
- *understand that the longitudinal component of tension $\tau_x$ is constant;*
- *be able to get $\tau_y$ knowing $\frac{\partial y}{\partial x}$;*
- *understand the part of the derivation where the "chunk" mass is $\mu\Delta x$;*
- *be able to convert expressions like $\frac{\frac{\partial y}{\partial x}(x+\Delta x,t)-\frac{\partial y}{\partial x}(x,t)}{\Delta x}$ to the proper partial derivatives, $\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}(x,t)\right)$, in the limit $\Delta x \to 0$;*
- *be able to confirm a given $y(x,t)$ as a solution to the wave equation;*
- *be able to derive dispersion relation $\omega = f(k)$ from the standing wave form $y(x,t) = A\cos(\omega t)\cos(kx)$;*

### 1. Problem 1:

In lecture we derived an expression for the complex amplitude of a driven and damped harmonic oscillator: $\underline{A} = Ae^{i\phi_0}$ with is the *real amplitude* $A$ and the *initial phase* $\phi_0$ given by

$$A = |\underline{A}| = \frac{f}{\sqrt{(\omega_0^2-\omega^2)^2+(b\omega)^2}} \qquad \text{and} \qquad \phi_0 = \tan^{-1}\left(\frac{b\omega}{\omega^2-\omega_0^2}\right) \quad , \tag{1}$$

where $f \equiv F_0/m$ is the amplitude of the driving force in units of mass and $\omega_0 \equiv \sqrt{k/m}$ is the natural frequency of the system.

In order to better appreciate the physical significance of these quantities and their frequency dependence, it is useful to *plot* them versus frequency and to identify certain important parameters of these plots. In this problem you will do this, starting from analyzing $(A/f)(\omega)$ and $\phi_0(\omega)$ in various limits[1]:

(a) Evaluate $(A/f)(\omega = 0)$ and $\phi_0(\omega = 0)$.

[1] We will be interested in the *normalized* plot $(A/f)(\omega)$ rather that in $A(\omega)$, because the former clearly shows how the amplitude of the driven oscillations compares with the amplitude of the driving force.

(b) Evaluate $\lim_{\omega\to\infty}(A/f)(\omega)$ and $\lim_{\omega\to\infty}\phi_o(\omega)$.

(c) Show that $(A/f)(\omega)$ has a *maximum* at $\omega = \omega_R = \sqrt{\omega_o^2 - b^2/2}$ if $b^2 < 2\omega_o^2$. Express the value of $(A/f)$ at the maximum in terms of $\omega_o$ and $b$.

*Hint:* Argue that the ratio of two positive expressions has a maximum when the denominator has a minimum. Then show that the denominator of $A$ in eq. (1) has a minimum at $\omega = \omega_R$ by checking that $\left.\frac{dD(\omega)}{d\omega}\right|_{\omega=\omega_R} = 0$ for the function $D(\omega) \equiv (\omega_o^2 - \omega^2)^2 + (b\omega)^2$.

(d) In the limit $b \ll \omega_o$, show that $\omega_R \approx \omega_o$ and that $(A/f)(\omega)$ is approximately equal to $(A/f)_{\max}/\sqrt{2}$ at each of the frequencies $\omega_{1,2} = \omega_o \pm b/2$. Compute the 'frequency band-width'[2] $\Delta\omega \equiv \omega_2 - \omega_1$.

(e) Sketch the plots of the functions $(A/f)(\omega)$ and $\phi_o(\omega)$ in the limit $b \ll \omega_o$. Label all important quantities, such as $(A/f)_{\max}$, $\Delta\omega$, etc., expressed in terms of $\omega_o$ and $b$.

**2. Problem 2:**

Many cordless phones transmit and receive 900 MHz microwaves. Find the wavelength $\lambda$, the wavenumber $\kappa$, and propagation number $k$. Microwaves travel through air at speed $c = 3.0 \times 10^8$ m/s.

**3. Problem 3:**

*Young & Freedman*, Problem 19-12.

**4. Problem 4:**

The propagation of waves along a stretched DNA molecule is described by the modified wave equation

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + a^2\frac{\partial^4 y}{\partial x^4} = 0 \tag{2}$$

where $c$ and $a$ are constants. ($c = \sqrt{\tau/\mu}$, where $\tau$ is the tension and $\mu$ the linear mass density as for the string considered in class, and $a$ is an elastic parameter independent of tension.)

Show that the standing wave $y(x,t) = A\cos(\omega t)\cos(kx)$ is a solution to the DNA wave equation (2) and derive the dispersion relation $\omega = f(k)$.

**5. Problem 5:**

A standing wave on a string of mass $m$ fixed at both ends ($x = 0$ and $x = L$) is described by

$$y(x,t) = A\sin(kx)\cos(\omega t)\ . \tag{3}$$

Assume that the amplitude of the wave is small ($kA \ll 1$) and make any justifiable approximations based on that assumption. Consider $m$, $A$, $k$, and $\omega$ to be known quantities (i.e. you may leave them in your answers).

(a) What is the $x$-component of the force due to the string on the fixed point $x = 0$? (Remember, the string is under tension so it pulls on whatever is holding it.)

[2] The frequency band-width is the range of frequencies 'preferred' by the driven/damped oscillator. The number $Q \equiv \frac{1}{bT} \approx \frac{1}{2\pi}\frac{\omega_0}{\Delta\omega}$ is called a 'quality factor' or a '$Q$-factor'. It is a measure 'how sharp' the resonance is. The value of $Q$ shows how good a mechanical system is in 'selecting' certain frequencies. Large $Q$-factors are important for instance in pieso-crystals setting the precise timing of quartz watches and computers.

(b) What is the $y$-component of the force due to the string on the fixed point $x = 0$?

(c) Consider a tiny *chunk* of string of length $dx$ between $x$ and $x + dx$. Find the $x$- and $y$-components of the force on this chunk due to the rest of the string on the left side only.

(d) Find the $x$- and $y$-components of the force on this chunk due to the rest of the string on the right side only.

(e) Find the net force on the chunk.

(f) Verify that $\vec{F} = m\vec{a}$ works for the chunk.

(g) Is the net force on the chunk proportional to its displacement from equilibrium? If so, what is the effective *spring constant* for the restoring force acting on the chunk?