SOLUTIONS TO PROBLEM SET #6

① a) The general solution to the wave equation can be written $y(x,t) = f_1(x-ct) + f_2(x+ct)$. The first condition implies $f_1(x) = -f_2(x)$, while the second condition $\frac{\partial y}{\partial t}\Big|_{t=0} = V_0 \cos(kx + \pi/3)$ implies that f_1 & f_2 are sinusoidal functions. Once the form of the function is known, one can guess at a solution, since x & t are always guaranteed to appear in the combination $x \pm ct$.

Since $\frac{\partial y}{\partial t} \sim$ cosine, $y(x,t) \stackrel{?}{=} (\quad)\sin(kx - \omega t + \pi/3) + (\quad)\sin(kx + \omega t + \pi/3)$. From boundary condition $\frac{\partial y}{\partial t}\Big|_{t=0} = V_0 \cos(kx + \pi/3)$, $(\quad) = \frac{V_0}{2\omega}$, so:

$y(x,t) = \frac{V_0}{2\omega}\sin(kx + \omega t + \pi/3) - \frac{V_0}{2\omega}\sin(kx - \omega t + \pi/3)$, where the sign has been chosen for the second term so that $y(x,0) = 0$. This $y(x,t)$ is of the form $f_1(x-ct) + f_2(x+ct)$, and satisfies the initial conditions, so it is the particular solution of the wave eq. for this problem.

b) Using the angle addition formulae for $\sin(A \pm B)$, $y(x,t)$ can be rewritten as $\frac{V_0}{\omega}\cos(kx + \pi/3)\sin\omega t$ which is in the standard form of a standing wave solution $y(x,t) = A(x)\sin\omega t$, with $A(x) = V_0/\omega \cos(kx + \pi/3) = \frac{V_0}{ck}\cos(kx + \pi/3)$, using the dispersion relation $\omega = ck$ for sinusoidal traveling waves.

(2) a) As in (1), $y(x,t) = f_1(x-ct) + f_2(x+ct)$.

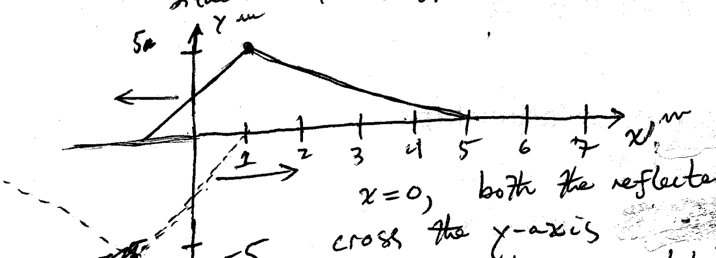
condition (1) says $y(x,0) = Ae^{-(x/a)^2}$, while condition (2) says $\frac{\partial y}{\partial t}\Big|_{t=0} = -\frac{2xcA}{a^2}e^{-(x/a)^2}$. The general solution says that x & t always appear in $y(x,t)$ as $x \pm ct$, so $y(x,t) \overset{?}{=} AC_1 e^{-(\frac{x-ct}{a})^2} + AC_2 e^{-(\frac{x+ct}{a})^2}$. Taking $\frac{\partial y}{\partial t} = +2C_1\left(\frac{x-ct}{a^2}\right)\frac{c}{a^2}Ae^{-(\frac{x-ct}{a})^2} - 2C_2\left(\frac{x+ct}{a^2}\right)\frac{c}{a^2}Ae^{-(\frac{x+ct}{a})^2}$, so $\frac{\partial y}{\partial t}\Big|_{t=0} = \frac{2C_1 c}{a^4}xAe^{-(x/a)^2} - \frac{2C_2 c}{a^4}xAe^{-(x/a)^2}$. Letting $C_1 = -\frac{a^2}{2} = -C_2$ gives $\frac{\partial y}{\partial t}\Big|_{t=0} = -\frac{2Acx}{a^2}e^{-(x/a)^2}$ as required.

(C_1 & C_2 adjustable constants)

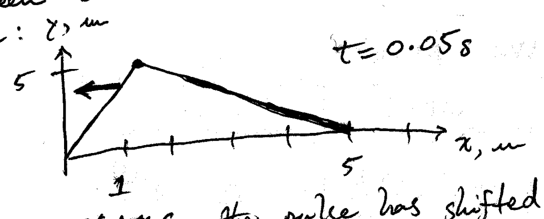
Then $y(x,t) = \frac{Aa^2}{2}e^{-(\frac{x+ct}{a})^2} - \frac{Aa^2}{2}e^{-(\frac{x-ct}{a})^2}$, and $y(x,0) = 0$, in contradiction to the condition (1). So both $f_1(x+ct)$ & $f_2(x-ct)$ cannot be present in the solution. Since $\frac{\partial y}{\partial t}\Big|_{t=0} = \ominus \frac{2xcA}{a^2}e^{-(x/a)^2}$, $y(x,t) \overset{?}{=} AC_1 e^{-((x+ct)/a)^2}$. Then $y(x,0) = C_1Ae^{-(x/a)^2} \Rightarrow C_1 = 1$, and $\frac{\partial y}{\partial t} = -2A\left(\frac{x+ct}{a^2}\right)c\,e^{-(\frac{x+ct}{a})^2} \Rightarrow \frac{\partial y}{\partial t}\Big|_{t=0} = -\frac{2Axc}{a^2}e^{-(x/a)^2}$. Since $y(x,t) = Ae^{-(\frac{x+ct}{a})^2}$ is of the form $y(x,t) = f_1(x+ct) + f_2(x-ct)$, and satisfies both ~~boundary~~ initial conditions, it is the particular solution to the problem.

b) This is clearly not a standing wave, but it does propagate due to the $x+ct$ dependence. A point on the wave is represented by a particular value of $y(x,t)$, and hence a particular value of the argument $\left(\frac{x+ct}{a^2}\right)$ in the exponent. As t increases, the x position representing a point on the wave must decrease for the argument to stay constant. Thus the wave travels towards $-x$.

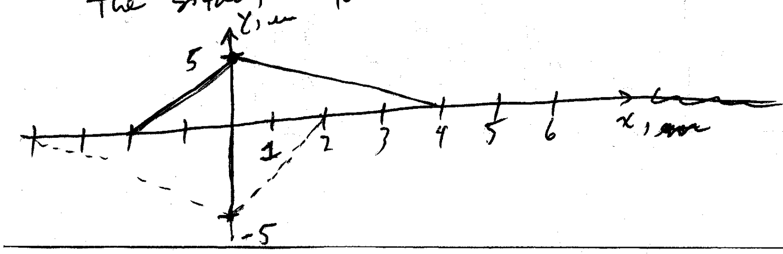
③ a) The pulse is moving towards $-x$. Thus, by the argument in problem ② part b), and the general solution to the wave equation, $y(x,t) = f_1(x+ct)$ for the pulse. The pulse ~~maintains~~ ~~is~~ ~~assumed~~ is assumed to maintain its shape, it is just shifted on the x-axis as t increases. Again, a point on the pulse has a particular value of ~~f~~. Increasing t by 0.05s implies that x must decrease by $100\,m/s \times 0.05s = 5m$. ~~this~~ $x=0$ represents a fixed support, so the boundary condition on the left is $y(0,t)=0$, so the general solution for the system $y(x,t) = f_1(ct+x) + f_2(ct-x)$ shows that $f_1(ct) = -f_2(ct)$; there is a reflected pulse traveling towards $+x$ that is inverted (out of phase) with the pulse traveling towards $-x$. At $t=0.05s$, the situation looks like:



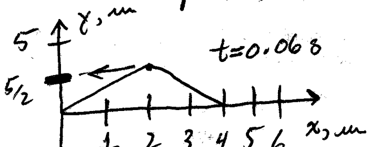
where the dashed line is the reflected pulse. Clearly, at $x=0$, both the reflected and the traveling pulse cross the y-axis with the same amplitude and opposite phase, and thus completely destructively interfere, as they must, since $x=0$ is a fixed end. Destructive interference between $0 < x < 1$ occurs, increasing the slope of the L.H.S. of the pulse:



At $t = 0.06\,s$, the pulse has shifted $100\,m/s \times 0.06s = 6m$. The situation looks like:



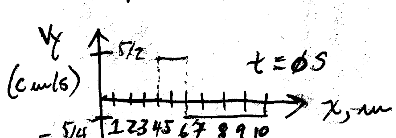
Again, at the fixed end there is complete destructive interference, and partial interference for $0<x<2$. In this region, $f_1 = 5 - \frac{5}{4}x$ (traveling pulse), while $f_2 = \frac{5}{2}x - 5$ (reflected pulse), so overall: $y = \frac{5}{4}x$, so:



where the figure is symmetric (even if it doesn't look that way), and the new maximum of $5/2$ was determined from the equation $f_1 = 5 - \frac{5}{4}x$ for the traveling pulse evaluated at $x=2$.

b) The velocity in the y direction of a point on the string is $\frac{\partial y}{\partial t} = \frac{\partial f_1}{\partial t} + \frac{\partial f_2}{\partial t}$. An important relationship for the functions f_1 & f_2 appearing in the general solution of the wave equation is: $\frac{\partial}{\partial t} f(x \pm ct) = \pm c \frac{\partial}{\partial x} f(x \pm ct)$, where c is the wave speed. Thus, $\frac{\partial y}{\partial t} = c\frac{\partial f_1}{\partial x} - c\frac{\partial f_2}{\partial x}$. The slopes $\frac{\partial f_1}{\partial x}$ & $\frac{\partial f_2}{\partial x}$ can be read from the graphs above, and inserted into the equation for $\frac{\partial y}{\partial t}$:

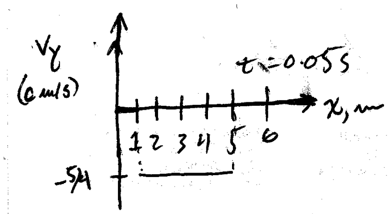
For $t=0$: $\frac{\partial f_1}{\partial x} = \begin{cases} -5/4 & 6<x<10 \\ 5/2 & 4<x<6 \\ 0 & \text{elsewhere} \end{cases}$, $\frac{\partial f_2}{\partial x} = 0$ in the region of interest. Thus, graphing in units of c m/s (100 m/s) for simplicity:



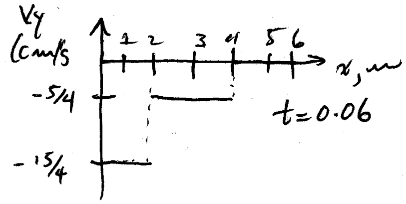
For $t=0.05$ s, the situation is complicated by the presence of the reflected pulse. In this case:

$\frac{\partial f_1}{\partial x} = \begin{cases} -5/4 & 1<x<5 \\ 5/2 & x>0, x<1 \\ 0 & \text{elsewhere} \end{cases}$, $\frac{\partial f_2}{\partial x} = \begin{cases} 5/2 & 0<x<1 \\ 0 & x>1 \end{cases}$, the only ~~area~~ intervals relevant to the problem. Then:

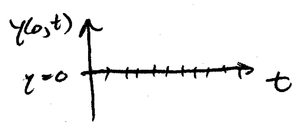
$V_y(x, 0.05\,s) = \begin{cases} -\frac{5}{4}c & 1<x<5 \\ \frac{5}{2}c - \frac{5}{2}c & 0<x<1 \end{cases}$



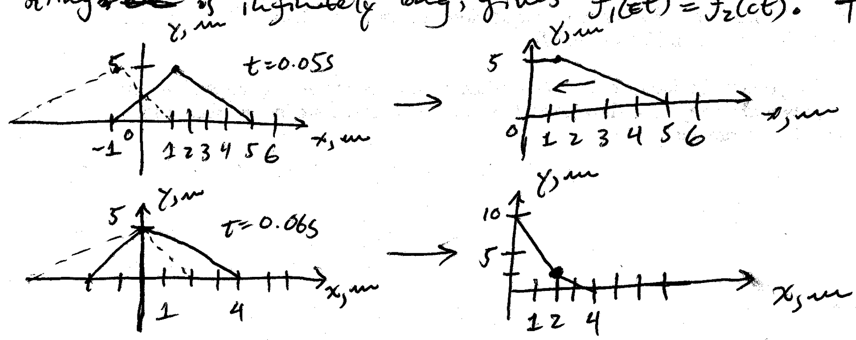
Similarly for $t=0.06$, $v_y = \begin{cases} -\frac{5}{4}c - \frac{5}{2}c & 0 < x < 2 \\ -\frac{5}{4}c & 2 < x < 4 \\ 0 & x > 4 \end{cases}$



c) For a fixed end at $t=0$, the sketch isn't very interesting:

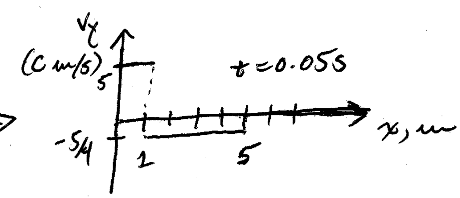


d) For a free end, the boundary condition is: $\left.\frac{\partial y}{\partial x}\right|_{x=0} = 0$. Thus, using the general solution, $\frac{\partial f_1(ct)}{\partial x} = \frac{\partial f_2(ct)}{\partial x} \Rightarrow$ the travelling & reflected pulses are in phase: $\frac{\partial}{\partial x}(f_1 - f_2) = 0$ $\Rightarrow f_1 - f_2 = \text{const.}$, where using the boundary condition that the string is infinitely long, gives $f_1(ct) = f_2(ct)$. Then:



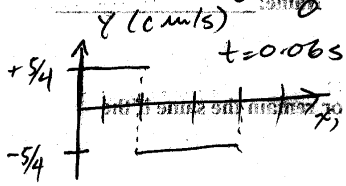
e) The velocity distributions can be calculated in the same way as in part b):

$v_y(x, 0.05\,s) = \begin{cases} -\frac{5}{4}c & 1 < x < 5 \\ 5c & 0 < x < 1 \\ 0 & x > 5 \end{cases} \Rightarrow$



$$v_y(x, 0.06s) = \begin{cases} -5/4\,c & 2<x<4 \\ -5/4 + \frac{5}{2}c & 0<x<2 \end{cases} \Rightarrow$$

$x > 4$

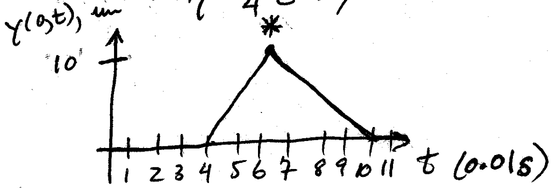


f) For a free end, the particle history is a little more interesting:

① $0<t<0.04$ $\quad y(0,t)=0$

$0.04<t<0.06$ the particle goes from 0 to 10 m with velocity $5c$ m/s

$0.06<t<0.10$ the particle goes from 10 to 0 m with velocity $\frac{10}{4}c$ m/s



* note that the velocity changes discontinuously at $t = 0.06s$, $x=0$.

④ a) Clearly, the function will have the same shape, but will shift 2 units in the $-u$ direction.

b) The function won't shift (the node at $u=0$ remains the same), but the decay will ~~cause~~ occur more rapidly, causing a ~~a~~ left-right contraction.

c) The exponential part is even, but the sine is odd, so switching the sign in the argument gives an overall up-down reflection. Furthermore, the decay occurs more slowly, so there will be a left-right expansion.

d) This is just a combination of the effects described in parts a) & b) — a shift 2 units to the right, and a left-right contraction.

⑤ a) Writing Newton's second Law for the ring:

$m_{ring}\ddot{y}(0,t) = -b\frac{\partial y}{\partial t}\Big|_{x=0} + \tau\frac{\partial y}{\partial x}\Big|_{x=0}$, where $\ddot{y}(0,t)$ is the second partial derivative of y w/ respect to t at $x=0$, where the ring & string are attached.

Since $m_{ring}=0$ by assumption:

$$-b\frac{\partial y}{\partial t}\Big|_{x=0} + \tau\frac{\partial y}{\partial x}\Big|_{x=0} = 0$$

is the required boundary condition.

b) Using the general solution $y(x,t) = f_1(ct-x) + f_2(ct+x)$, the boundary condition reads:

$-b\left(\dot{f}_1(ct) + \dot{f}_2(ct)\right) + \tau\left(f_1'(ct) + f_2'(ct)\right)$, where a $\cdot$ denotes a time partial derivative, and a $'$ denotes a space partial derivative. Again using the relation in problem ② part b), $f_1'(ct) = -\frac{1}{c}\dot{f}_1(ct)$, and $f_2'(ct) = \frac{1}{c}\dot{f}_2(ct)$

Then the boundary condition becomes:

$$-\left(+b+\frac{\tau}{c}\right)\dot{f}_1(ct) + \left(\frac{\tau}{c} - b\right)\dot{f}_2(ct) = 0$$

If $b = \frac{T}{c}$, then $\dot{f}_1(ct) = 0$, implying that f_1 is constant in time at $x=0$. But because of the form of the solution $f_1 = f_1(ct - x)$, the only way this is possible is if f_1 does not propagate. If, at say $t=0$, when the student initiates the pulse, the string is undisturbed, then $f_1(ct-x) \equiv 0$, so there can be no reflected pulse! For any other value, b gives information about the relative amounts & phase of f_1 & f_2 in the solution, but does not inhibit reflection. Thus, the required value for b is: $b = \frac{T}{c}$. Since waves on a string propagate at a speed $\sqrt{\frac{T}{\mu}} = c$, $b = \sqrt{T\mu}$.