P214 Solutions

Problem Set 7

1) Phase Diagrams

Interference: $I_N = |\text{sum of wave amplitudes}|^2 = \left|\sum_{i=1}^{N} \underline{A}_i\right|^2$

$\underline{A}_i$ = amplitude of i^{th} wave

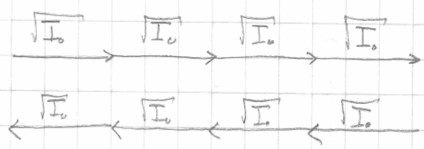
in polar coordinates: $\underline{A}_i = A_i e^{i\phi_i}$

From Class: $I_4 = |A_1 + A_2 e^{ik\Delta R} + A_3 e^{i2k\Delta R} + A_4 e^{i3k\Delta R}|^2$

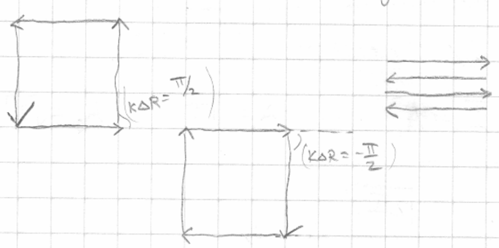
If they all have the same amplitude, A_0 $\quad I_0 = A_0^2$, $A_0 = \sqrt{I_0}$

$$\Rightarrow I_4 = I_0 |1 + e^{ik\Delta R} + e^{i2k\Delta R} + e^{i3k\Delta R}|^2$$

(a) Maximum for 4 slits:



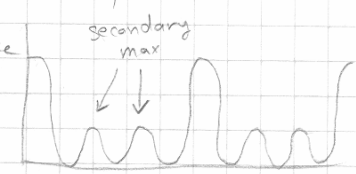
(b) Minimum for 4 slits: will give 0 intensity



(c) In class we derived a general expression for the intensity:

$$I = I_0 \frac{\sin^2\left(\frac{Nk\Delta r}{2}\right)}{\sin^2\left(\frac{k\Delta r}{2}\right)}$$

Look like



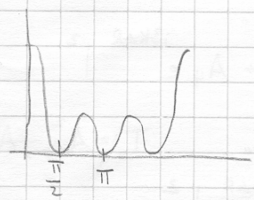
We can approximate the 2nd secondary max as being $\frac{1}{2}$ between the minima.

The minima will occur when the numerator $\longrightarrow 0$
[provided the denominator doesn't go to 0]

so we want $\sin^2 \frac{Nk\Delta r}{2} = 0 \quad (N=4)$

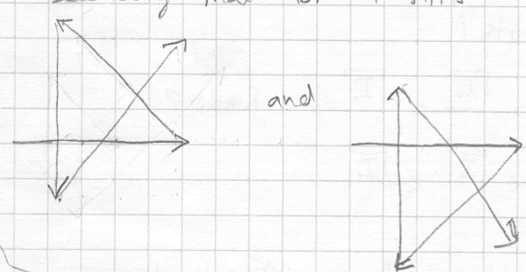
$\Rightarrow \quad 2k\Delta r = m\pi \qquad \Delta r = d\sin\theta$

$\Rightarrow \quad k\Delta r = \frac{m\pi}{2}$



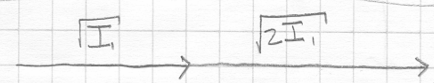
so $k\Delta r$ for max $\approx \pm\frac{3\pi}{4}$

Secondary Max for 4 slits:



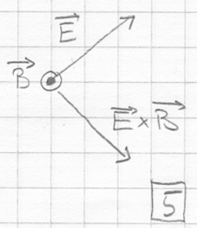
(d) Max for 2 slits w/ intensities $I_2 = 2I_1$

$I_2 = A_2^2, \quad I_1 = A_1^2 \quad$ so $A_1 = \sqrt{I_1}, \; A_2 = \sqrt{2}\sqrt{I_1}$



2) (a) From the directions of the $\vec{E}$ and $\vec{B}$ fields we can determine the direction of propogation:

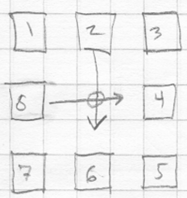
$\vec{E} \times \vec{B}$ gives direction of propogation.



(using right hand rule)

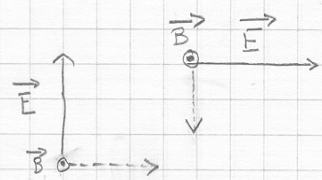
So the wave is heading towards $\boxed{5}$ which means it $\boxed{\text{originated from } \boxed{1}}$

(b) We want to set up 2 transmitters:



We still want the $\vec{E}$ field in the plane of the page.

The following combo works:



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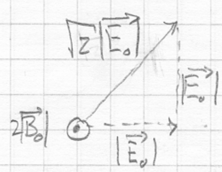
-----> (direction of propogation)

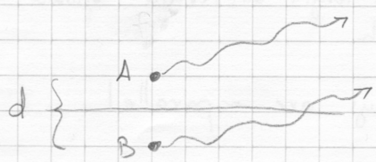
(c) The resultant fields will add at the center.



From one source we know that $|\vec{E}| = c|\vec{B}|$, but this relation won't hold for the sum of our 2 fields.

$$\left(\frac{|\vec{E}_0|}{|\vec{B}_0|} = c, \text{ but: } \frac{\sqrt{2}|\vec{E}_0|}{2|\vec{B}_0|} = \frac{c}{\sqrt{2}} \right)$$

3) Two speakers are emitting waves with the same frequencies and phases



The two waves will interfere with each other.
Remember, interference is just the sum of the wave amplitudes squared!

$$I = \left| A_1 e^{i(kr_1+\phi_1)} + A_2 e^{i(kr_2+\phi_2)} \right|^2$$

$$= \left| e^{ikr_1+i\phi_1} \right|^2 \left| A_1 + A_2 e^{ik(r_2-r_1)+i(\phi_2-\phi_1)} \right|^2$$

When the phases are the same $(\phi_2-\phi_1)=0$

$$\Rightarrow I = \left| A_1 + A_2 e^{ik\Delta R} \right|^2$$

The intensity for a single slit is: $I_1 = A_1^2$

$$\Rightarrow A_1 = \sqrt{I_1}$$

so $\boxed{I = \left| \sqrt{I_A} + \sqrt{I_B}\, e^{ik\Delta r} \right|^2}$

Now we're ready to go!

(a) $k = \frac{2\pi}{\lambda}$ $\quad c = \lambda f \quad \Rightarrow \lambda = \frac{c}{f}$

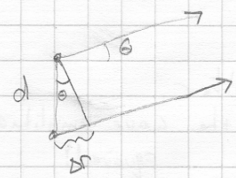
so $\boxed{k = \frac{2\pi f}{c}}$

(b) A maximum will occur when the Path Length Difference is a multiple of the wavelength.

so $\Delta r = m\lambda$ where m is an integer.

Note: The intensities from each source aren't the same but a maximum will still occur when both waves "reinforce" each other. So the path length difference should still be a multiple of the wavelength.

When the wall is far away the rays are ~parallel



$$\Delta r = d\sin\theta_{max} = m\lambda$$

So $\boxed{\sin(\theta_{max}) = m\frac{\lambda}{d}}$

$\boxed{\theta_{max} = \sin^{-1}\left(\frac{m\lambda}{d}\right)}$

So $\Delta r = m\lambda$ or $k\Delta r = 2\pi m$ $\quad e^{i2\pi m} = +1$

$$I = \left|\sqrt{I_A} + \sqrt{I_B}\, e^{ik\Delta r}\right|^2$$

$$I_{max} = \left|\sqrt{I_A} + \sqrt{I_B}\right|^2 = I_A + I_B + 2\sqrt{I_A}\sqrt{I_B}$$

$$= (16 + 25 + 2\cdot 4\cdot 5)\ W/m^2$$

$$\Rightarrow \boxed{I_{max} = 81\ \frac{W}{m^2}}$$

(c) Minima should occur when the Path Length Difference is a half multiple of the wavelength.

$$\Rightarrow \Delta r = \left(\frac{2m+1}{2}\right)\lambda = d\sin\theta_{min}$$

So $\boxed{\theta_{min} = \sin^{-1}\left[\left(\frac{2m+1}{2}\right)\frac{\lambda}{d}\right]}$

$$k\Delta r = (2m+1)\pi$$

$$I_{min} = \left|\sqrt{I_A} - \sqrt{I_B}\right|^2 = I_A + I_B - 2\sqrt{I_A}\sqrt{I_B}$$

$$= (16 + 25 - 2\cdot 4\cdot 5)\ W/m^2$$

$$\boxed{I_{min} = 1\ W/m^2}$$

(d) We found a formula for the intensity in the beginning of the problem:

$$I = \left|\sqrt{I_A} + \sqrt{I_B}\, e^{ik\Delta r}\right|^2$$

$$= \left(\sqrt{I_A} + \sqrt{I_B}\, e^{ik\Delta r}\right)\left(\sqrt{I_A} + \sqrt{I_B}\, e^{-ik\Delta r}\right)$$

$$= I_A + I_B + \sqrt{I_A}\sqrt{I_B}\left(e^{ik\Delta r} + e^{-ik\Delta r}\right)$$

note: $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ so $e^{ik\Delta r} + e^{-ik\Delta r} = 2\cos(k\Delta r)$

so $I = I_A + I_B + 2\sqrt{I_A}\sqrt{I_B}\cos(k\Delta r)$

$\Delta r = d\sin\theta$

$$\Rightarrow \boxed{I(\theta) = \left(41 + 40\cos[kd\sin\theta]\right)\, W/m^2}$$

(e) If we introduced a phase shift we could have written:

$$I = \left|\sqrt{I_A} + \sqrt{I_B}\, e^{i(k\Delta r + \Delta\phi)}\right|^2 \quad \text{where } \Delta\phi = \phi_B - \phi_A$$

A max. will occur when $k\Delta r + \Delta\phi = 0$

so $\Delta\phi = -k\Delta r = -kd\sin\theta$

For $k = 5m^{-1}$ $d = 2m$, $\theta = 10^\circ$ $\Delta\phi = -10\sin(10^\circ) = -1.736^r$
$\cong -100^\circ$

$\boxed{\phi_B - \phi_A \cong -100^\circ} = -1.74^r$

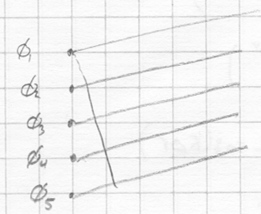
The phase of B should lead the phase of A since
$\phi_B = \phi_A - 100^\circ$

(Think of B being shifted forward.)
$f(x-b)$

There fore the $\boxed{\text{phase of A should Lag that of B.}}$

4) We have 5 radio towers emitting radio waves equally spaced apart.

This is analogous to 5 slits each of which has a different phase.



The intensity will be the sum of the amplitudes squared.

$$I = \left| \sqrt{I_1}\, e^{i(kr_1+\phi_1)} + \sqrt{I_2}\, e^{i(kr_2+\phi_2)} + \sqrt{I_3}\, e^{i(kr_3+\phi_3)} + I_4\, e^{i(kr_4+\phi_4)} + \sqrt{I_5}\, e^{i(kr_5+\phi_5)} \right.$$

The amplitudes are all the same so we can factor them out.

I'm also going to factor out $\left|e^{i(kr_1+\phi_1)}\right|^2 = 1$

$$\Rightarrow I = I_0 \left| 1 + e^{i[k\Delta r+(\phi_2-\phi_1)]} + e^{i[2k\Delta r+(\phi_3-\phi_1)]} + e^{i[3k\Delta r+(\phi_4-\phi_1)]} + e^{i[5k\Delta r+(\phi_5-\phi_1)]} \right|$$

Notice $\phi_2-\phi_1 = 1\cdot\frac{\pi}{2}$ $\quad \phi_3-\phi_1 = 2\cdot\frac{\pi}{2}$ $\quad \phi_4-\phi_1 = 3\cdot\frac{\pi}{2}$ $\quad \phi_5-\phi_1 = 4\cdot\frac{\pi}{2}$

let $\Delta\phi = \frac{\pi}{2}$

$$I = I_0 \left| 1 + e^{i[k\Delta r+\Delta\phi]} + e^{2i[k\Delta r+\Delta\phi]} + e^{3i[k\Delta r+\Delta\phi]} + e^{4i[k\Delta r+\Delta\phi]} \right|^2$$

In class we had the same result for N slits but with $\Delta\phi = 0$

The result was $I = I_0 \dfrac{\sin^2\left(\frac{Nk\Delta R}{2}\right)}{\sin^2\left(\frac{k\Delta R}{2}\right)}$

For our case $k\Delta R \rightarrow k\Delta R + \Delta\phi$

so $\boxed{I = I_0 \dfrac{\sin^2\left(\frac{Nk\Delta R}{2} + \frac{N\Delta\phi}{2}\right)}{\sin^2\left(\frac{k\Delta R}{2} + \frac{\Delta\phi}{2}\right)}}$ $\quad N = 5$

(a) The path length difference will be 0 for $\theta = 0$ since all the rays are ~parallel.

so $\boxed{k\Delta R = 0}$

We can plug this into our equation for the intensity:

$$I(\theta=0) = I_0 \frac{\sin^2\left(\frac{5\pi}{4}\right)}{\sin^2\left(\frac{\pi}{4}\right)} = I_0$$

This is not an interference minimum but corresponds to a secondary maximum. (First secondary max is between the minima at $\frac{k\Delta R}{2} = \frac{\pi}{5}, \frac{2\pi}{5}$. So the max is close to $\frac{3\pi}{10} \approx \frac{\pi}{4}$.)

(b) The principle maxima will occur when the denominator goes to 0.

$$\Rightarrow \sin^2\left(\frac{k\Delta R}{2} + \frac{\Delta\phi}{2}\right) = 0$$

so $$\frac{k\Delta R}{2} + \frac{\Delta\phi}{2} = m\pi$$

$$\Rightarrow k\Delta R + \Delta\phi = 2\pi m \qquad \Delta\phi = \frac{\pi}{2}, \quad \Delta R = d\sin\theta$$

$$\frac{2\pi d \sin\theta}{\lambda} + \frac{\pi}{2} = 2\pi m$$

$$2d\sin\theta = \left(2m - \frac{1}{2}\right)\lambda, \quad m = 0, \pm1, \pm2, \ldots$$

$$\Rightarrow \sin\theta_m = \left(m - \frac{1}{4}\right)\frac{\lambda}{d}$$

$$\boxed{\theta_1 = \sin^{-1}\left(\frac{3\lambda}{4d}\right) = \sin^{-1}\left(\frac{3\cdot 100}{4\cdot 300}\right) = 0.253^r}$$

(c) We just derived our general expression for θ,

$$\theta = \sin^{-1}\left[\left(m - \frac{1}{4}\right)\frac{\lambda}{d}\right] = \sin^{-1}\left[\left(m - \frac{1}{4}\right)\frac{1}{12}\right]$$

$$m = 0, \pm1, \pm2, \ldots$$