Physics 214 **Fall 2001**

Problem Set # 8

Reading on current material and for coming lectures: *Young & Freedman*, Sections 38.1—38-6 (Diffraction); Sections 19.8, 33.5 (Energy and Momentum).

Problems: (due 11/15)

Skills to be mastered:

- *Be able to use the complex representation to study more complicated solutions to the wave equation, e.g., superpositions of plane waves of different wavevectors;*
- *Be able to study superposition of waves of different frequencies;*
- *Be able to write down the general solution of the wave equation in the vicinity of an interface between two media of different wave speeds;*
- *Be able to derive the amplitude of the superposition of an incident and multiple reflected waves, using the 'history-of-events' approach the general solution of the wave equation in the vicinity of an interface between two media of different wave speeds;*
- *Be able to derive expressions for the diffraction pattern intensity of a wide slit in different situations.*

1. Problem 1 (Beating):

Consider the two speakers (A and B) of *PS#7, Problem 3* (separation d, distance $R \gg d$ from the dormitory wall where people are listening to the transmission.) We are going to consider a room located at $\theta = 0$, known as the *"Room of the Beats"* (see Figure 1).

Assume now that the **intensities** of both speakers emit are the **same**, but their **frequencies** ω_1 and ω_2 are **different**. As a result the sound waves from the speakers, a distance R away, are of the form:

$$S_i(R,t) = S_0 e^{-ik_iR} e^{i\omega_i t} \;, \tag{1}$$

where $i = 1, 2$ labels the speaker from which the wave is coming (note that S_0 is assumed to be the same for both waves).

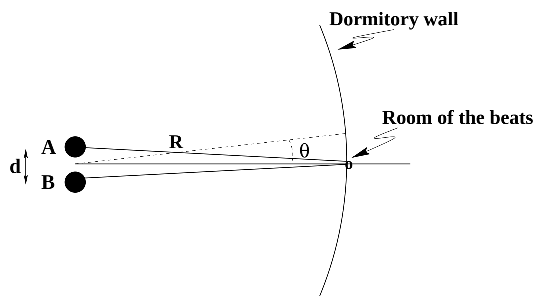

Figure 1: Beating of speaker sounds.

(a) ***Derive an expression for the sound displacement field*** $S(R,t)$ ***in the Room of the Beats*** $(\theta = 0)$.

Hint: $S(R,t) = S_1(R,t) + S_2(R,t)$, where $S_1(R,t)$ and $S_2(R,t)$, given by eq. (1), are the sound displacements in the room caused by each speaker separately.

(b) ***Sketch*** $S(R,t)$***.*** Be sure to specify the frequencies of the slowly varying and rapidly varying part of the the function and to understand how they should be represented on the plot.

2. Problem 2 (Plane Waves with Different Wavevectors, Group Velocity):
The function $y(x,t)$ is a sum of two plane waves of the same (real) amplitude A:

$$y(x,t) = \Re\mathfrak{e}\left[Ae^{i(k_1 x - \omega_1 t)} + Ae^{i(k_2 x - \omega_2 t)}\right] . \tag{2}$$

(a) ***Show*** that $y(x,t)$ ***satisfies the wave equation*** for a string. ***What is the dispersion relation?***

(b) ***Simplify the expression*** (2) and ***write it as a product of trigonometric functions***.

Hint: Use complex algebra and the trigonometric identity

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) . \tag{3}$$

(c) ***What is the condition*** for $y(x,t)$ to have a ***maximum?*** Using this condition, ***show that the maxima travel with velocity*** $v_g = \dfrac{\Delta\omega}{\Delta k}$, where $\Delta\omega = \omega_2 - \omega_1$ and $\Delta k = k_2 - k_1$. This is called **group velocity**.

Hint: You can either analyze your expression of part (b) or, better yet, write down the condition which the phases (i.e., the exponents) of the two plane waves in (2) have to satisfy if they are to add constructively.

(d) If $k_1 = 9k_0$ $k_2 = 11k_0$, ***what is the group velocity? Use this to sketch*** $y(x,t)$ at $t = 0$ ***and*** $t = 2\pi/\omega_0$.

(e) The plane waves are also solutions to the more complicated wave equation of *PS#3, Problem 4* (dispersion relation $\omega = \omega(k) = ck\sqrt{1+a^2k^2}$ with $a = 1/10k_0$). Using the same k_1 and k_2 as in (d), ***find*** ω_1 ***and*** ω_2 ***in this case. Calculate the group velocity.***

3. Problem 3 (Interface between Different Media):
In this problem we are going to study the solution to the wave equation for sound near the interface between gaseous media, with densities ρ' and ρ and bulk moduli B' and B respectively. (You can think of the interface as a **massless** membrane that separates the two gases.) The origin of the coordinate system $x = 0$ is chosen to be at the interface and there are no other interfaces or boundaries around (see Figure 2):

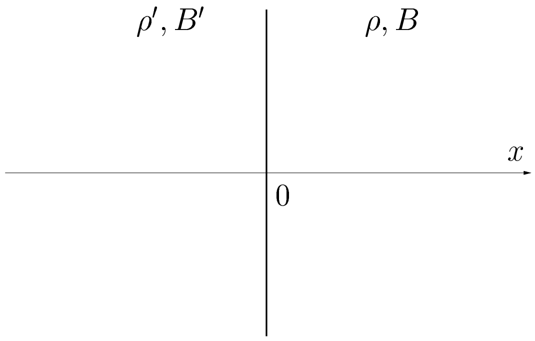

Figure 2: Interface between different media.

(a) ***Show*** that the ***boundary conditions*** for the displacement field at the interface are:

$$\begin{aligned} S'(x=0,t) &= S(x=0,t) \ ; \\ B'\frac{\partial S'}{\partial x}(x=0,t) &= B\frac{\partial S}{\partial x}(x=0,t) \ , \end{aligned} \tag{4}$$

where $S'(x,t)$ is the solution to the wave equation for sound in the *left* $(x<0)$ region and $S(x=0,t)$ the solution in the *right* $(x>0)$ region.

(c) Assume the following general solutions to the sound wave equations in the two regions:

$$\begin{aligned} S'(x,t) &= h(x+c't) \ ; \\ S(x,t) &= f(x+ct)+g(x-ct) \ , \end{aligned} \tag{5}$$

where $c' = \sqrt{B'/\rho'}$ and $c=\sqrt{B/\rho}$ are the speeds of sounds in the two gases. Applying the boundary conditions (4), ***show that the functions*** g ***and*** h ***are related to*** f ***by:***

$$\begin{aligned} h\left(\frac{c'}{c}u\right) &= \frac{2}{1+\alpha}f(u) \ ; \\ g(-u) &= \frac{1-\alpha}{1+\alpha}f(u) \ ; \end{aligned} \tag{6}$$

where $\alpha \equiv \sqrt{(\rho' B')/(\rho B)}$. ***Show that*** (6) ***is consistent*** with the *open* and *closed* boundary conditions for f and g ***in the limits*** $\alpha \to 0$ ***and*** $\alpha \to \infty$***.***

(b) Now consider a plane wave, $f(x+ct) = \Re\mathfrak{e}\left[Ae^{i(kx+\omega t)}\right]$, incident on the interface. ***Show that the amplitude*** $A(x)$***, of the superposition*** of incident and reflected waves, for any $x>0$ is:

$$A(x) = A\left|1+e^{-ikx}(\pm r)e^{-ikx}\right| \ , \tag{7}$$

where $r \equiv |(1-\alpha)/(1+\alpha)|$ and the $\pm$ sign depends on whether the boundary is 'soft' $(\alpha<1)$ or 'hard' $(\alpha>1)$.

4. Problem 4 (Reflections from a Slab):

Now consider a plane wave of visible light in air $(n \approx 1)$ moving in the negative x direction. The wave is incident on a slab of glass of thickness d and index of refraction $n' > n$. As a result of reflections from both surfaces of the glass, the solution to the wave equation in the region $x>0$ will be a superposition

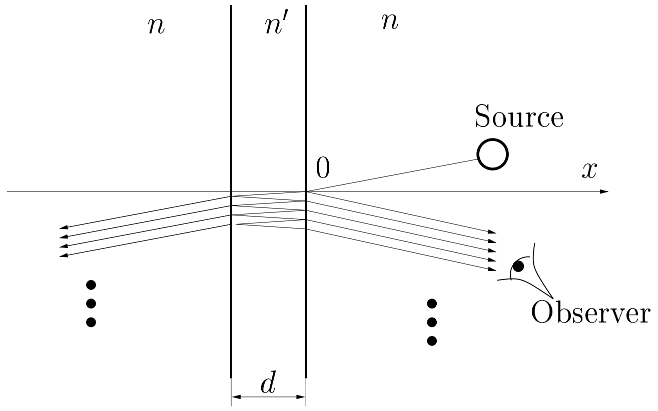

Figure 3: Multiple reflections from a slab of glass.

of the incident wave and the reflected waves. **Consider almost normal incidence,** i.e., assume all the rays on Figure 3 to be **perpendicular to the surface of the slab.**

The formula (7) of the previous problem, for the amplitude of the superposition, can be generalized to account for this more complicated case, in the following fashion:

$$A(x) = \left| \sum_{\text{paths}} \left[\prod_i a_i \right] \right| \tag{8}$$

In the above expression we are summing over **all possible paths** that light can take and each **path** is represented as a **product of events** ($\prod_i a_i$). An **'event'**, a_i, can be a propagation phase factor (e.g., e^{-ikx}), a reflection from an interface (a factor of $\pm r$), or a transmission through an interface (a factor of t or t'). For example, the first path is just 1 (light coming to the eye from the source nearby); the second path is $e^{-ikx}(-r)e^{-ikx}$ (the ray reflected from the front surface of the slab); etc: $A(x) = \left|1 + e^{-ikx}(-r)e^{-ikx} + \ldots\right|$.

More specifically, a *reflection* from a 'hard' boundary ($n' > n$) is represented by an ($-r$), while a reflection from a 'soft' boundary ($n' < n$) is represented by an ($+r$) in the path history $\prod_i a_i$. The transmission from air to glass is represented by t and from glass to air by t', where t, t', and r satisfy the relation $tt' = 1 - r^2$.

As in the previous problem, r, t, and t' depend on the properties of the two media: $r \equiv |(1-\alpha)/(1+\alpha)|$, $t \equiv 2/(1+\alpha)$, and $t' \equiv 2\alpha/(1+\alpha)$, where $\alpha \equiv c/c' = n'/n$ is the analogous quantity for light to the 'α' of Problem 3.

(a) Considering **multiple** reflections and using eq. (8), ***show that the amplitude*** $A(x)$, of the superposition wave at any $x > 0$, ***is given by the expression:***

$$A(x) = \left|1 + e^{-ikx} R(d) e^{-ikx}\right| , \qquad R(d) = -r \left[\frac{1 - e^{-2ikd}}{1 - r^2 e^{-2ikd}} \right] , \tag{9}$$

(b) ***Compute*** $\lim_{d\to\infty} R(d)$ and ***compare*** the result with eq. (7). ***What is*** $\lim_{d\to 0} R(d)$***?*** Does the result make physical sense?

5. Problem 5 (Wide-Slit Diffraction with Off-Axis Source):
A laser beam (wavelength λ) is shone on a **wide slit** (width $a = \lambda/2$), so that the angle between the beam and the normal to the slit plate is $\alpha = 30°$ (see Figure 4).

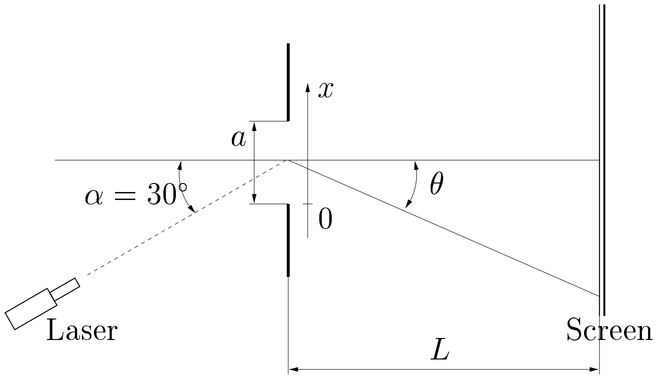

Figure 4: Wide slit with an off-axis source.

(The laser is far away from the slit, so you can assume that the light arrives there in the form of **plane waves**.)

(a) Because the light waves from the laser have to travel different distances to each point of the slit, the points along the slit act like **point sources with different phases**. ***Find the phase*** $\phi(x)$ of a spherical wavelet emerging from a point at a distance x from the bottom of the slit, relative to the point at $x = 0$.

(b) ***Derive a formula for the intensity*** $I(\theta)$ of the diffraction pattern formed on a screen placed at a distance $L \gg a$ behind the slit.

Hint: Split the finite slit into N infinitesimal slits distance d apart (i.e., at positions $x = 0, d, 2d, \ldots$) and derive an N-slit interference formula similar to the one in *Problem 5* of the *Exercise Problems*. Then take the limit $N \to \infty$, $d \to 0$ with $Nd = a$, as in lectures.

(c) ***Identify the locations*** of the ***principal maximum*** and ***all the minima*** and ***sketch a plot of the diffraction pattern*** on the screen.