11

(1) Using the general form given we can write the sound waves emitted from each speaker in the following way:

$$S_A(R,t) = S_o \, \mathrm{Re}\left[e^{-ik_A R} e^{i\omega_A t}\right] \qquad S_B(R,t) = S_o \, \mathrm{Re}\left[e^{-ik_B R} e^{i\omega_B t}\right]$$

at $\theta = 0$, the sound displacement is the sum of these two expressions

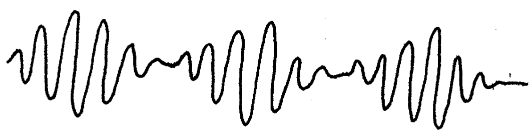
$$S(R,t) = S_A(R,t) + S_B(R,t) = S_o \, \mathrm{Re}\left[e^{-ik_A R} e^{i\omega_A t} + e^{-ik_B R} e^{i\omega_B t}\right]$$

now for simplicity let $-k_A R = \varphi_A$ and $-k_B R = \varphi_B$

using the trig identity: $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$$S(R,t) = 2S_o \cos\left(\frac{\omega_A+\omega_B}{2} t + \frac{\varphi_A+\varphi_B}{2}\right)\cos\left(\frac{\omega_A-\omega_B}{2} t + \frac{\varphi_A-\varphi_B}{2}\right)$$

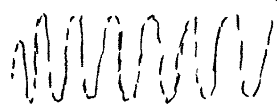
b) this is an expression for beats which looks like:



The frequency of the slow varying part (the packet):

is $\omega_A - \omega_B$

The frequency of the rapidly varying part:



is $\omega_A + \omega_B$

(2) a) wave eq.: $\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$

$$\frac{\partial^2}{\partial x^2} y(x,t) = \mathrm{Re}\, A\left[-k_1^2 e^{i(k_1 x - \omega_1 t)} + -k_2^2 e^{i(k_2 x - \omega_2 t)}\right]$$

$$\frac{\partial^2}{\partial t^2} y(x,t) = \mathrm{Re}\, A\left[-\omega_1^2 e^{i(k_1 x - \omega_1 t)} - \omega_2^2 e^{i(k_2 x - \omega_2 t)}\right]$$

Now plug these into wave equation

$$-k_1^2 \,\mathrm{Re}\, A\left[e^{i(k_1 x - \omega_1 t)}\right] - k_2^2 \,\mathrm{Re}\, A\left[e^{i(k_2 x - \omega_2 t)}\right] \stackrel{?}{=} \frac{1}{c^2}\left\{-\omega_1^2 \,\mathrm{Re} A\left[e^{i(k_1 x - \omega_1 t)}\right] - \omega_2^2 \,\mathrm{Re} A\left[e^{i(k_2 x - \omega_2 t)}\right]\right.$$

For this to be true, coefficients on like exponentials must be equivalent

$$-k_1^2 = \frac{-\omega_1^2}{c^2} \quad \text{and} \quad -k_2^2 = \frac{-\omega_2^2}{c^2}$$

$\therefore$ dispersion relations are: $k_1 = \frac{\omega_1}{c}$ and $k_2 = \frac{\omega_2}{c}$

② b) using Euler's Identity:

$$y(x,t) = \text{Re}\, A\left[\cos(k_1x-\omega_1t) + i\sin(k_1x-\omega_1t) + \cos(k_2x-\omega_2t) + i\sin(k_2x-\omega_2t)\right]$$

now take out the Real part: $y(x,t) = A\left[\cos(k_1x-\omega_1t) + \cos(k_2x-\omega_2t)\right]$

using trig id. given:

$$y(x,t) = 2A\cos\left[\underbrace{\left(\frac{k_1+k_2}{2}\right)}_{k_+}x - \underbrace{\left(\frac{\omega_1+\omega_2}{2}\right)}_{\omega_+}t\right]\cos\left[\underbrace{\left(\frac{k_1-k_2}{2}\right)}_{k_-}x - \underbrace{\left(\frac{\omega_1-\omega_2}{2}\right)}_{\omega_-}t\right]$$

$$\Rightarrow y(x,t) = 2A\cos(k_+x-\omega_+t)\cos(k_-x-\omega_-t)$$

c) In order for phases to add constructively: $k_1x - \omega_1 t - k_2x + \omega_2 t = n2\pi$

$$n2\pi = (k_1-k_2)x + (\omega_2-\omega_1)t \qquad \text{let } \Delta\omega \equiv \omega_2-\omega_1$$

$$n2\pi = -\Delta k x + \Delta\omega t \qquad \Delta k \equiv k_2 - k_1$$

$$\Delta\omega t = \Delta k x + n2\pi$$

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{x}{t} + \frac{n2\pi}{t\Delta k} \quad \therefore \quad x_m(t) = \frac{2n\pi}{\Delta k} + v_g t$$

Maxima travel in space (as time evolves) with speed $v_g = \frac{\Delta\omega}{\Delta k}$

d) $\Delta\omega = \omega_2 - \omega_1 = c\,11k_0 - c9k_0 = 2ck_0$

$\Delta k = k_2 - k_1 = 11k_0 - 9k_0 = 2k_0$

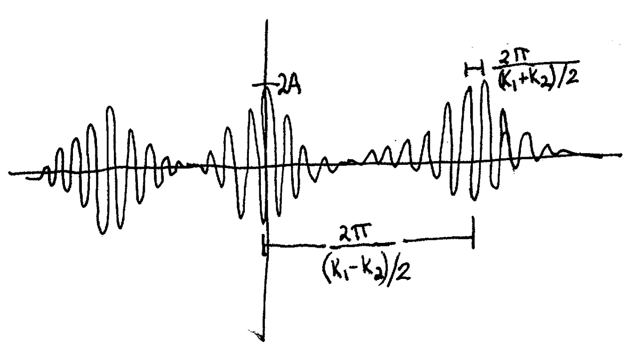
using $\omega_1 = ck_1$ and $\omega_2 = ck_2$ which also holds for $\frac{\Delta\omega}{\Delta k} = c = v_g$

group velocity: $\frac{\Delta\omega}{\Delta k} = \frac{2ck_0}{2k_0} = \frac{x}{t} + \frac{n2\pi}{2k_0t}$

$$c = \frac{x}{t} + \frac{n\pi}{k_0t} \implies x_m(t) = ct + \frac{n\pi}{k_0}$$

at $\boxed{t=0}$ $y(x,t) = 2A\cos(10k_0(x-ct))\cos(-k_0(x-ct))$

$y(x,0) = 2A\cos(10k_0x)\cos(-k_0x)$



at $\boxed{t=\frac{2\pi}{\omega_0}}$

$$x_m(t) = ct + \frac{n\pi}{k_0}$$

$$= c\frac{2\pi}{\omega_0} + \frac{n\pi}{k_0}$$

$$= \frac{(2+n)\pi}{k_0}$$

$$k_0x = (2+n)\pi$$

the graph will look exactly the same because it will shift by an integer multiply of π, put maxima in the same place as before

② continued

e) $\omega(k) = ck\sqrt{1+a^2k^2}$

$\omega_1(k_1) = c\,9k_0\sqrt{1+\frac{1}{100k_0^2}81k_0^2}$ $\qquad$ $\omega_2(k_2) = c\,11k_0\sqrt{1+\frac{1}{100\cancel{k_0^2}}121\cancel{k_0^2}}$

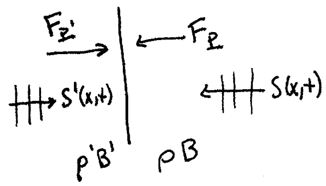
$\omega_1 = c\,9k_0\sqrt{\frac{181}{100}}$ $\qquad$ $\omega_2 = c\,11k_0\sqrt{\frac{221}{100}}$

$\omega_1 = ck_0\frac{9}{10}\sqrt{181} \approx 12.11\,ck_0$ $\qquad$ $\omega_2 = ck_0\frac{11}{10}\sqrt{221} \approx 16.35\,ck_0$

$\Rightarrow \Delta\omega \cong 4.24\,ck_0 \Rightarrow v_g = \frac{\Delta\omega}{\Delta k} \cong 2.12c \neq c.$

③ a) $\sum F = ma \Rightarrow F_P - F_{P'} = ma$ look at forces on membrane

massless membrane



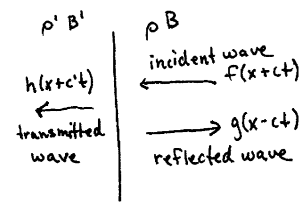
$AP - AP' = \cancel{ma}$

$A\left(P_0 - B\frac{\partial s}{\partial x}(x,t)\right)\Big|_{x=0} = A\left(P_0 - B'\frac{\partial s'}{\partial x}(x,t)\right)\Big|_{x=0}$

$$B\frac{\partial s(x,t)}{\partial x}\Big|_{x=0} = B'\frac{\partial s'(x,t)}{\partial x}\Big|_{x=0}$$

Also $s'(x=0,t) = s(x=0,t)$

this is because the displacement fields must be equal at the interface in order to avoid discontinuity



b) using BC: $s'(0,t) = s(0,t)$

$h(c't) = f(ct) + g(-ct)$ $\qquad$ let $u = ct$

$h\left(\frac{c'}{c}u\right) = f(u) + g(-u)$

using BC: $B'\frac{\partial}{\partial x}s'\Big|_{x=0} = B\frac{\partial}{\partial x}s\Big|_{x=0}$

1st find $\frac{\partial}{\partial x}s$ $\qquad$ $B'\frac{\partial}{\partial x}h(x+c't) = B\left(\frac{\partial}{\partial x}f(x+ct) + \frac{\partial}{\partial x}g(x-ct)\right)$

From pulse eq. we know any solution will obey relation $\frac{\partial s}{\partial t} = \pm c\frac{\partial s}{\partial x}$

$\therefore$ we get $\frac{B'}{c'}\frac{\partial}{\partial t}h(x+c't) = \frac{B}{c}\left(\frac{\partial}{\partial t}f(x+ct) - \frac{\partial}{\partial t}g(x-ct)\right)$

set $x=0$ $\qquad$ $\frac{B'}{c'}\frac{\partial}{\partial t}h(c't) = \frac{B}{c}\frac{\partial}{\partial t}f(ct) - \frac{B}{c}\frac{\partial}{\partial t}g(-ct)$

let $u=ct$ and divide by $\frac{B}{c}$ $\qquad$ note that $\frac{B'}{c'} = \frac{B'}{\sqrt{\frac{B'}{\rho'}}} = \frac{B'}{\sqrt{B'}}\sqrt{\rho'} = \sqrt{B'\rho'}$ $\quad$ likewise $\frac{B}{c} = \sqrt{B\rho}$

③ continued

we can define $\alpha \equiv \sqrt{\frac{\rho' B'}{\rho B}}$

and we now have: $\alpha \dot{h}(\frac{c'}{c}u) = \dot{f}(u) - \dot{g}(-u)$

now integrate with respect to time: $\alpha h(\frac{c'}{c}u) = f(u) - g(-u)$

now use relation from 1st BC: $\alpha(f(u) + g(-u)) = f(u) - g(-u)$

$\Rightarrow \alpha f(u) + \alpha g(-u) = f(u) - g(-u)$

$f(u)(1-\alpha) = g(-u)(\alpha+1) \Longrightarrow \boxed{g(-u) = \frac{1-\alpha}{\alpha+1} f(u)}$

now go back to 1st relation: $h(\frac{c'}{c}u) = f(u) + g(-u)$ and replace $g(-u)$

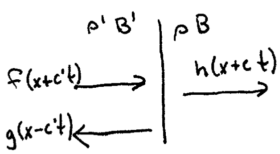
$h(\frac{c'}{c}u) = f(u) + \frac{1-\alpha}{\alpha+1} f(u)$ find common denominators

$$\boxed{h(\frac{c'}{c}u) = \frac{2}{\alpha+1} f(u)}$$

ASIDE helpful for problem 4:

In problem 4 we let $r \equiv \frac{1-\alpha}{\alpha+1}$ and $t \equiv \frac{2}{1+\alpha}$

Note that if we exchange the media: $(B\rho) \leftrightarrow (B'\rho')$ which means incident and reflected waves $f(u)$ & $g(-u)$ are in primed material and transmitted wave is in unprimed material



$\alpha \to \frac{1}{\alpha}$

which means $r' = \frac{1-\frac{1}{\alpha}}{\frac{1}{\alpha}+1} = \frac{\frac{\alpha-1}{\alpha}}{\frac{\alpha+1}{\alpha}} = \frac{\alpha-1}{\alpha+1} = -r$

also $t' = \frac{2}{\frac{1}{\alpha}+1} = \frac{2}{\frac{\alpha+1}{\alpha}} = \frac{2\alpha}{1+\alpha}$ and $\therefore tt' = 1 - r^2$

---

back to 3b)

Now look at extreme cases:

as $\alpha \to 0$ $\frac{1-\alpha}{1+\alpha} \to 1$ and we get $g(-u) = f(u)$

this makes sense as an open boundary condition because the reflected wave is not inverted

as $\alpha \to \infty$ $\frac{1-\alpha}{1+\alpha} \approx -\frac{\alpha}{\alpha} = -1$ $g(-u) = -f(u)$

which makes sense as a closed boundary condition because the reflected wave is inverted

(3) continued

c) $f(x+ct) = \mathrm{Re}\left[A e^{i(kx+\omega t)}\right]$

$S(x,t) = f(x+ct) + g(x-ct)$

using relations found in b) we can derive $g(x-ct)$ from $f(x+ct)$

$$g(-u) = \frac{1-\alpha}{1+\alpha} f(u) \Rightarrow g(-u) = \frac{1-\alpha}{1+\alpha} \mathrm{Re}\left[A e^{i(ku)}\right]$$

$$\therefore g(-(x-ct)) = \frac{1-\alpha}{1+\alpha} \mathrm{Re}\left[A e^{i(k(-(x-ct)))}\right] = \frac{1-\alpha}{1+\alpha} \mathrm{Re}\left[A e^{i(-kx+\omega t)}\right]$$

let $r \equiv \frac{1-\alpha}{1+\alpha}$ $\quad S(x,t) = \mathrm{Re}\left[A e^{i(kx+\omega t)} + r A e^{i(kx+\omega t)}\right]$

$$S(x,t) = \mathrm{Re}\left[e^{i\omega t} \underbrace{\left[A e^{ikx} + A r e^{-ikx}\right]}_{\underline{A}(x)}\right] = \mathrm{Re}\left[\underline{A}(x) e^{i\omega t}\right]$$

look closer at $\underline{A}(x)$

$\underline{A}(x) = A\left(e^{ikx} + r e^{-ikx}\right)$ because we want a real # when looking at amplitude, find magnitude of $\underline{A}(x)$

$|\underline{A}(x)| = |A| \left|e^{ikx} + r e^{-ikx}\right|$

A is a real # $\therefore$ $|A| = A$

factor out e^{ikx}

$$|\underline{A}(x)| = A \left|e^{ikx}\right| \left|1 + \frac{r e^{-ikx}}{e^{ikx}}\right| = \boxed{A \left|1 + r e^{-ikx} e^{-ikx}\right|}$$

(4) $A(x) = \left| \sum_{paths} \left[\prod_i a_i \right] \right|$ the amplitude is the sum of products of events. There is a product of events for every path

path 1: incident light gives value of 1 as in equation #5

path 2: reflected light off of 1st interface gives:

n' n

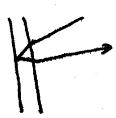
$$e^{-ikx}(-r)\,e^{-ikx}$$

e^{-ikx} ← represents travel away from glass back to source/observer

e^{-ikx}: represents travel to glass from source/observer

$(-r)$: reflection from interface. Negative because $n' > n$

path 3: reflected light off of 2nd interface gives:



$\leftrightarrow$ d

$$e^{-ikx}(t)\,e^{-ikd}\,(+r)\,e^{-ikd}\,(t')\,e^{-ikx} = tt' r\, e^{-2ikx}\, e^{-2ikd}$$

(t): transmission through 1st interface (air to glass)

e^{-ikd}: travel through width of glass

$(+r)$: reflection off of 2nd interface

(t'): transmission through 1st interface (glass to air)

path 4: reflects 2 times off of 2nd interface

$\leftrightarrow$ d

$$e^{-ikx}\, t\, e^{-ikd}\,(r)\,e^{-ikd}\, r\, e^{-ikd}\, r\, e^{-ikd}\, t'\, e^{-ikx}$$

$$= tt' r^3 e^{-2ikx} e^{-4ikd}$$

You should now see a pattern for further paths, sum these to get $A(x)$

$$A(x) = A\left| 1 + -r e^{-2ikx} + tt' r e^{-2ikx} e^{-2ikd} + tt' r^3 e^{-2ikx} e^{-4ikd} + \dots \right|$$

rearrange...

$$A(x) = A\left| 1 - r e^{-2ikx} \underbrace{\left\{ 1 - tt' e^{-2ikd}\left[1 + r^2 e^{-2ikd} + r^4 e^{-4ikd} + r^6 e^{-6ikd} + \dots \right] \right\}}_{-R_d/r} \right|$$

4 continued

we then see that $\left[1+r^2e^{-2ikd}+r^4e^{-4ikd}\ldots\right] = \dfrac{1}{1-r^2e^{-2ikd}}$

$$-\frac{R_d}{r} = 1 - \frac{tt'e^{-2ikd}}{1-r^2e^{-2ikd}} = \frac{1-r^2e^{-2ikd} - tt'e^{-2ikd}}{1-r^2e^{-2ikd}}$$

$$R_d = -r\left[\frac{1-(r^2+tt')e^{-2ikd}}{1-r^2e^{-2ikd}}\right] = -r\left[\frac{1-e^{-2ikd}}{1-r^2e^{-2ikd}}\right]$$

note: $r^2+tt' = 1$

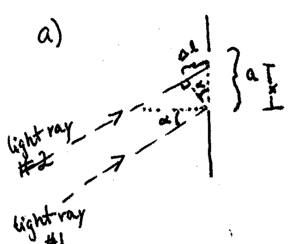
Check this by looking at the situation when $d\to\infty$. This corresponds to an infinitely wide piece of glass, or just an interface as in problem 3.

$R_d \xrightarrow[d]{\infty} -r$ which matches our results from problem 3.

$R_d \xrightarrow[d]{0} 0$ this makes sense because at $d\to 0$ we no longer have an interface and all light is transmitted, ie. there are no reflections.

5 Because the laser is very far away, light rays from laser entering the slits can be approximated as parallel.

a)



using trigonometry we see that $\Delta l = x\sin\alpha$

let phase at $x=0$ (the bottom of the slit) be zero

then $\boxed{\varphi(x) = kx\sin\alpha}$ where $0\le x\le a$ and $k=\frac{2\pi}{\lambda}$

b) $$I = \left|\sqrt{I_1}\,e^{i(kr_1+\varphi_1)} + \sqrt{I_2}\,e^{i(kr_2+\varphi_2)} + \ldots \sqrt{I_N}\,e^{i(kr_N+\varphi_N)}\right|^2$$

Divide slit up into N infinitesimal slits distance d apart.

$I_1 = I_2 = I_3 = \ldots I_N \equiv I_0$ factor out $e^{i(kr_1+\varphi_1)}$

$$I = I_0\left|e^{i(kr_1+\varphi_1)}\right|^2\left|1+e^{ik(r_2-r_1)}e^{i(\varphi_2-\varphi_1)} + \ldots e^{ik(r_N-r_1)}e^{i(\varphi_N-\varphi_1)}\right|^2$$

let $r_2-r_1 \equiv \Delta r$ $\therefore r_3-r_1 = 2\Delta r$ etc.

Also, from part a) we see that $\varphi_1 = k(0)\sin\alpha = 0$ $\varphi_2 = kd\sin\alpha$ and $\varphi_3 = k(2d)\sin\alpha = 2\varphi_2$ etc.

⑤ continued

Our intensity equation then becomes:

$$I = I_o(1)\left|1 + e^{i(K\Delta r + \varphi_2)} + e^{2i(K\Delta r + \varphi_2)} + \dots e^{(N-1)i(K\Delta r + \varphi_2)}\right|^2$$

noticing that this is a geometric series, we can rewrite this:

$$I = I_o\left|\frac{1 - e^{Ni(K\Delta r + \varphi_2)}}{1 - e^{i(K\Delta r + \varphi_2)}}\right|^2$$

finding the square magnitude + using the trig. ID as in lecture we get: $(\cos 2A = 1 - 2\sin^2 A)$

$$I = I_o \frac{\sin^2\left[\frac{N}{2}(K\Delta r + \varphi_2)\right]}{\sin^2\left[\frac{1}{2}(K\Delta r + \varphi_2)\right]}$$

now substitute $Nd = a$ as in lecture

remember that $K\Delta r = kd\sin\theta$ and $\varphi_2 = kd\sin\alpha$

$$\therefore \quad I = I_o \frac{\sin^2\left[\frac{Ka}{2}(\sin\theta + \sin\alpha)\right]}{\underbrace{\sin^2\left[\frac{Ka}{N2}(\sin\theta + \sin\alpha)\right]}}$$

or with new substitution

$K\Delta r = \frac{Ka}{N}\sin\theta$

$\varphi_2 = \frac{Ka}{N}\sin\alpha$

Now let $N \to \infty$

if N is very large, this is very small $\therefore$ use small angle approximation $\sin\theta \approx \theta$

$$I = \lim_{N\to\infty} \frac{\sin^2\left[\frac{Ka}{2}(\sin\theta + \sin\alpha)\right]}{\left(\frac{Ka}{N2}(\sin\theta + \sin\alpha)\right)^2} = I_{max} \frac{\sin^2\left[\frac{Ka}{2}(\sin\theta + \sin\alpha)\right]}{\left[\frac{Ka}{2}(\sin\theta + \sin\alpha)\right]^2}$$

c) look at limiting cases.

Principal max occur when denominator $\to 0$

$\therefore \frac{Ka}{2}(\sin\theta + \sin\alpha) = 0$ or $\sin\theta = -\sin\alpha = -\frac{1}{2}$

minima occur when numerator $\to 0$

$\therefore \frac{Ka}{2}(\sin\theta + \sin\alpha) = n\pi$ or $\sin\theta = \frac{2n\pi}{Ka} - \frac{1}{2}$ $\quad n \neq 0$