1. These are all right-moving pulses, so they satisfy the pulse equation:

$\frac{\partial y}{\partial t} = -c\frac{\partial y}{\partial x}$ $\quad c = \sqrt{\frac{\tau}{\mu}} = 3\,\frac{m}{s}$ $\quad \tau = 9\,N$ $\quad \mu = 1\,\frac{kg}{m}$

The equations we need are:

$ke(x) = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 \quad pe(x) = \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 \quad p(x) = -\tau\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}$

We need to manipulate $ke(x)$ so we can use it. Substitute the pulse equation to remove $\frac{\partial y}{\partial t}$:

$ke(x) = \frac{1}{2}\mu\left(-c\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}\mu c^2\left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}\mu\left(\frac{\tau}{\mu}\right)\left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 = pe(x)$

So in the region of each pulse, $ke = pe = \frac{9}{2}(\text{slope})^2$, and is constant.

By definition, $e(x) = ke(x) + pe(x)$, so $e = 9(\text{slope})^2$.

Because the energy density is constant in the region of each pulse, the total energy is:

$e_{tot} = \int_0^{12} e(x)\,dx = 9(\text{slope})^2(\text{width of pulse})$

	A	B	C	D	
slope	1	½	2	⅓	in $\frac{cm}{m}$
width	4	8	2	12	in m

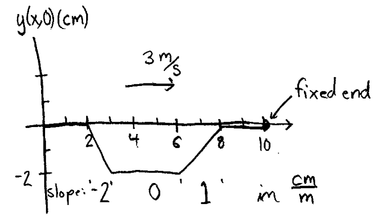
	ke	pe	e	e_{tot}
A	9/2	9/2	9	36
B	9/8	9/8	9/4	18
C	18	18	36	72
D	½	½	1	12

Units: Because the slopes are in $\frac{cm}{m}$, the units are somewhat strange. Energy densities have units of $\frac{kg \cdot cm^2}{s^2 m} = 10^{-4}\frac{J}{m}$.

Energy has units of 10^{-4} J.

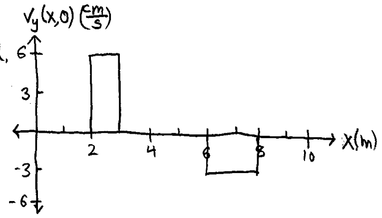
C has the highest total energy.

2. $\tau = 9\,N \quad \mu = 1\,\frac{kg}{m} \quad c = 3\,\frac{m}{s}$

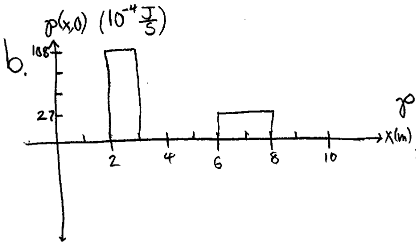


Pulse eqn. (right) $\quad v_y = -c(\text{slope})$

a.



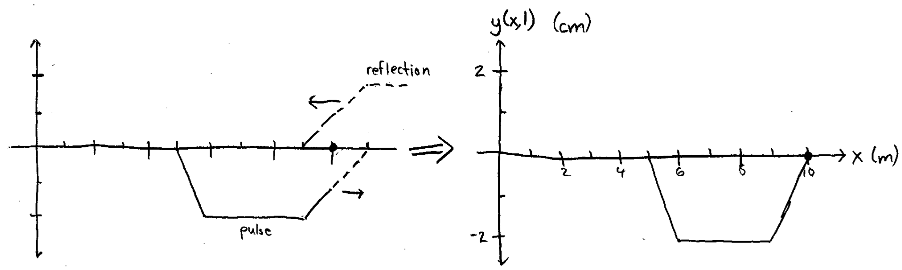
b.



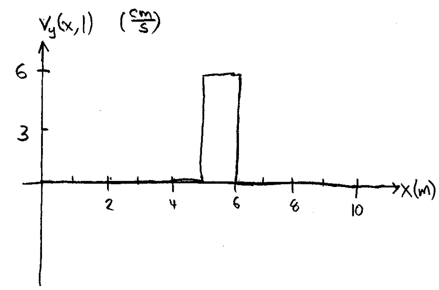
$$p = -\tau \frac{\partial y}{\partial t}\frac{\partial y}{\partial x}$$

$$= -9(v_y)(\text{slope})$$

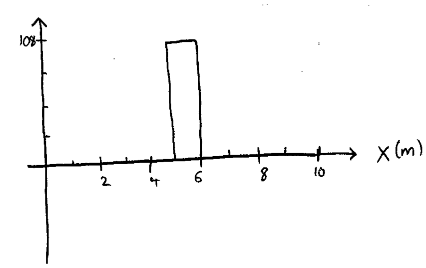
c.



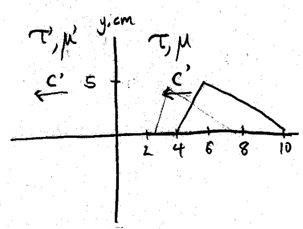
d.



e



Velocity is zero from 9 to 10 because the pulse and its reflection have the same slope, but move in opposite directions.

3. 

$c = \sqrt{\frac{\tau}{\mu}} = 100\,\frac{m}{s}$

$\mu' = 9\mu \Rightarrow c' = \frac{1}{3}c = \frac{100}{3}\,\frac{m}{s}$

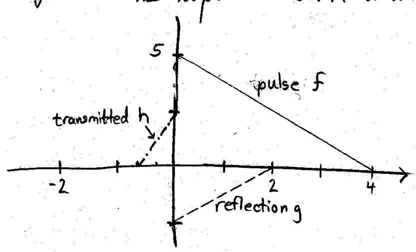
$\alpha = \frac{\sqrt{\tau'\mu'}}{\sqrt{\tau\mu}} = 3$ because $\tau' = \tau$

From PS 8, problem 3:

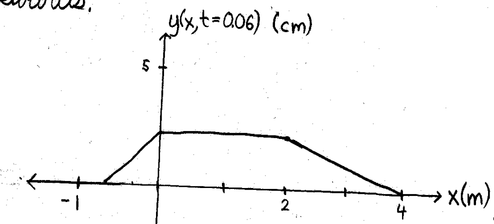
$h\left(\frac{c'}{c}u\right) = \frac{2}{1+\alpha} f(u) \qquad g(-u) = \frac{1-\alpha}{1+\alpha} f(u) \qquad \alpha = 3$

Rearranging: $h(u) = \frac{1}{2} f(3u) \qquad g(u) = -\frac{1}{2} f(-u)$

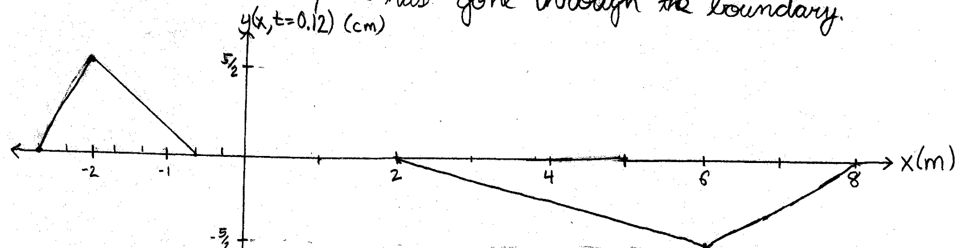
a. $t = 0.06$ s The peak of the pulse has reached the boundary. The front edge reached the boundary at $t = 0.04$, then moved $\left(\frac{100}{3}\right)(0.02) = \frac{2}{3}$ m. The reflection is upside-down and backwards.



$\Rightarrow$



$t = 0.12$ s The entire pulse has gone through the boundary.



Total energy: $e_{tot} = \int e(x)\,dx = \int \tau\left(\frac{\partial u}{\partial x}\right)^2 dx$

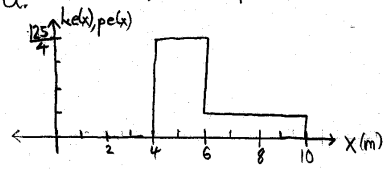
Incoming $e_{tot} = \left(\tau\left(\frac{5}{2}\right)^2\right)\cdot 2 + \left(\tau\left(\frac{5}{4}\right)^2\right)\cdot 4 = \frac{75\tau}{4}$ in units of 10^{-4} J

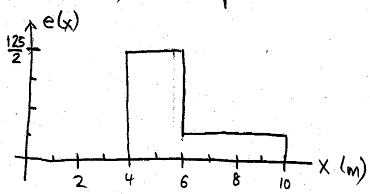
Reflected $e_{tot} = \left(\tau\left(\frac{5}{4}\right)^2\right)\cdot 2 + \left(\tau\left(\frac{5}{8}\right)^2\right)\cdot 4 = \frac{75\tau}{16}$

Transmitted $e_{tot} = \left(\tau\left(\frac{15}{4}\right)^2\right)\cdot\frac{2}{3} + \left(\tau\left(\frac{15}{8}\right)^2\right)\cdot 4 = 3\cdot\frac{75\tau}{16}$

Total energy in the reflected and transmitted waves is $(3+1)\cdot\frac{75\tau}{16} = \frac{75\tau}{4}$, the same as the energy from the incoming wave.

4a. As in (1), $ke(x) = pe(x) = \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2$ and $e(x) = ke(x) + pe(x) = \tau\left(\frac{\partial y}{\partial x}\right)^2$

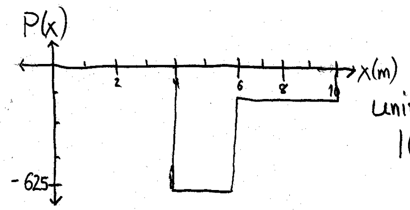




units: $10^{-4}\,\frac{J}{m}$
energy density

$P(x) = -\tau \frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = -\tau c\left(\frac{\partial y}{\partial x}\right)^2$

$\tau = 10\,N \quad \mu = .001\,\frac{kg}{m} \quad c = 100\,\frac{m}{s}$



units: $10^{-4}\,\frac{J}{s}$

b. $P(x) = -\tau c\left(\frac{\partial y}{\partial x}\right)^2 = -c\,e(x)$

The power is the energy times the speed of the wave.

c. Fixed end: $y(x=0,t) = 0 \Rightarrow \frac{\partial y(0,t)}{\partial t} = 0$

$P(0) = -\tau \frac{\partial y(0,t)}{\partial x}\overset{0}{\cancel{\frac{\partial y(0,t)}{\partial t}}} = 0$

The power is zero because no energy is leaving the string. After reflection, the wave is the same size and shape, so it has the same energy. Energy is conserved by a fixed end.

d. Free end: $\frac{\partial y(x=0,t)}{\partial x} = 0$

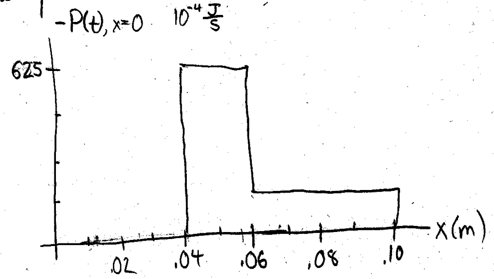
$P(0) = -\tau \overset{0}{\cancel{\frac{\partial y(0,t)}{\partial x}}}\frac{\partial y(0,t)}{\partial t} = 0$

Again, energy is conserved, and the reflection is the same size and shape.

5. For the power as a function of time, at the endpoint, we can find the power of the part of the wave that is arriving at the endpoint at that time.

We can just redraw the graph of the power, and label the horizontal axis with the time that each part of the pulse hits the endpoint.

$c = 100\ \frac{m}{s}$



b. $P_d = F_y v_y \qquad F_y = -b v_y \qquad v_y = \frac{\partial y(0,t)}{\partial t} = +c\frac{\partial y(0,t)}{\partial x}$

$$P_d = -b v_y^2 = -b c^2 \left(\frac{\partial y(0,t)}{\partial x}\right)^2 = -\left(\frac{\tau}{c}\right) c^2 \left(\frac{\partial y}{\partial x}\right)^2 = -\tau c \left(\frac{\partial y}{\partial x}\right)^2$$

The power dissipated by the dashpot is the same as the energy carried by the pulse.

This makes sense because the pulse is swallowed by the absorber. Energy leaves the string, and the only place it could have gone is to be dissipated by the dashpot.