ANSWERS

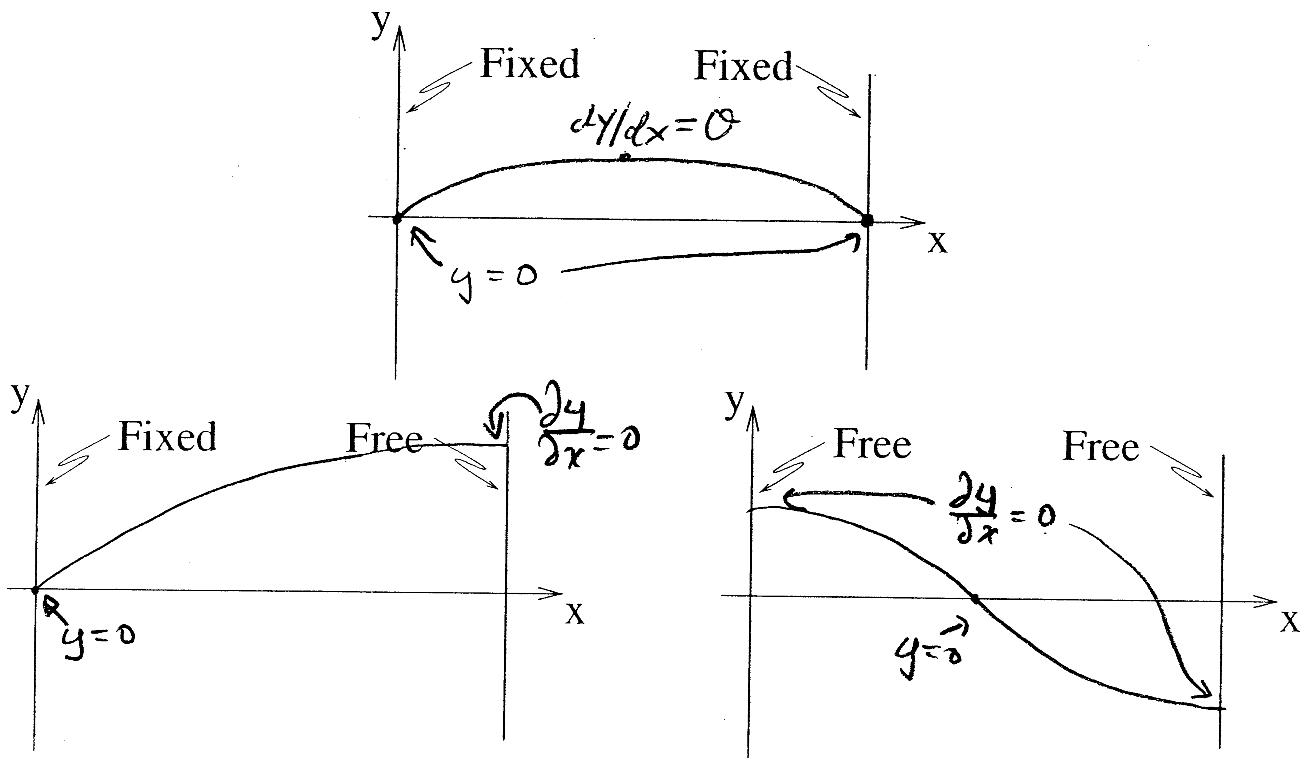

Figure 1: Wave patterns for different boundary conditions.

1 Problem 1: Standing Waves on Strings [16 points]

(a) Sketches (12 points)

On the axes which Figure 1 provides, ***sketch the form of the lowest frequency mode*** for each of the three possible combinations of boundary conditions (fixed–fixed, fixed–free, free–free).

(b) Comparing frequencies (4 points)

Among the three possible combinations of boundary conditions from part (a), which allows the lowest possible standing-wave frequency for the same length L, tension τ and mass M of the string?
Provide your answer in the box below.

$\boxed{\text{Fixed-Free}}$

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2 Problem 2: Lab Experiment I [20 points]

As part of the first laboratory experiment in Phys 214, you measured the frequency of the lowest mode of an aluminum rod of length $L = 1$ m. You should have found a frequency near $f = 2\times10^3$ Hz. (For the purpose of this problem, take this to be the exact value.) Because you held the rod at its center while leaving both ends free, this gives the frequency corresponding to the free–free combination of boundary conditions from Problem 3. Finally, further measurements on the actual rod you used show it to have a cross sectional area $A = \pi r^2 = 500$ mm^2 $= 500 \times 10^{-6}$ m^2 and a total mass $M = 1.35$ kg.

(a) Speed of sound in aluminum (8 points)

Use the above information to compute the speed of sound in the rod. Give a numerical result in units of m/s.

$$c = \lambda f = (2L)f = 2 \cdot 1\,\text{m} \cdot 2\times10^3\,\text{s}^{-1}$$

$$\boxed{c = 4000\ \text{m/s}}$$

(Note: From the figure, we see $\frac{\lambda}{2} = L \Rightarrow \lambda = 2L$.)

(b) Bulk modulus of aluminum (8 points)

The actual speed of sound in Aluminum is $c = 5100$ m/s. (**Note:** You will not find this value from the data in part (a) because of the fake value given there for f!) ***Using the actual speed of sound $c = 5100$ m/s and the information given above, compute the bulk modulus B of aluminum. Give a numerical result in units of N/m^2.***

$$c = \sqrt{\frac{B}{\rho_0}} \Rightarrow B = \rho_0 c^2$$

Need, $\rho_0 = \frac{M}{V} = \frac{M}{A \cdot L} = \frac{1.35\,\text{kg}}{500 \cdot 10^{-6}\,\text{m}^2 \cdot 1\,\text{m}} = 2700\ \frac{\text{kg}}{\text{m}^3}$

Thus, $B = 2700\ \frac{\text{kg}}{\text{m}^3}(5100\ \text{m/s})^2 = 70.2\times10^9\ \frac{\text{kg}\cdot\text{m}/\text{s}^2}{\text{m}^2}$

$$= \boxed{70.2\times10^9\ \frac{\text{N}}{\text{m}^2} = B}$$

(c) Spring constant of the rod (4 points) *Challenge Problem!*

The rod can be used as a very stiff spring. When the rod is compressed by a distance x, it pushes back with a restoring force proportional to x, $F = kx$. Use the definition

$$\Delta P = -B\frac{\Delta V}{V_0},$$

to ***compute a numerical value for the proportionality constant k in units of N/m.***
Hint: For this problem, you may ignore any tendency for the radius of the rod to increase as the rod is compressed. The influence of this effect is small in aluminum.

$$\Delta F = \Delta P \cdot A = -B\frac{\Delta V}{V_0}\cdot A = -B\frac{\cancel{A}\cdot \Delta x}{\cancel{A}\cdot L}\cdot A$$

$$\Rightarrow k = \frac{BA}{L} = \frac{70.2\times 10^9 \frac{N}{\cancel{m^2}}\cdot 500\times 10^{-6}\,\cancel{m^2}}{1\,m}$$

$$\boxed{k = 35\cdot 10^6\ \frac{N}{m}}$$

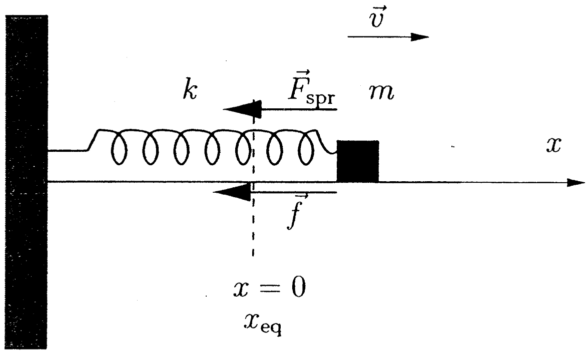


Figure 2: A mass-spring realization of a harmonic oscillator with sliding friction.

3 Problem 3: Harmonic Oscillator with Friction[24 points]

Consider a harmonic oscillator with mass m acted on by an ideal spring of spring constant k and equilibrium position $x = x_{eq} = 0$ *and* by a sliding friction force in the direction opposite to the motion and of magnitude $f = \mu m g$ where μ is the coefficient of sliding friction and g the acceleration of gravity. (See Figure 2.)

(a) Equation of Motion (6 points)

Show that, when the mass moves* to the right, *the Equation of Motion for the harmonic oscillator with friction is:

$$\frac{d^2x}{dt^2} + \mu g + \omega_0^2 x = 0\ , \qquad \omega_0 \equiv \sqrt{\frac{k}{m}}\ . \tag{3.1}$$

$$\sum F_x^{ext} = m a_x$$

$$-\overset{\omega_0^2}{\frac{k}{m}}(x - \overset{0}{x_{eq}}) - \mu \cancel{m} g = \cancel{m}\frac{d^2x}{dt^2}$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \mu g + \omega_0^2 x = 0, \quad \omega_0 \equiv \sqrt{\frac{k}{m}}}$$

(b) General solution (6 points)

Show that the following is a general solution to the equation of motion (3.1):

$$x(t) = -\frac{\mu g}{\omega_0^2} + \Re\mathfrak{e}\left[\underline{A}e^{i\omega_0 t}\right]. \tag{3.2}$$

Note: Be sure to list *each* requirement for a general solution as you check it.

1 Solves Eq.:

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\mathrm{Re}\left[\underline{A}\, i\omega_0 e^{i\omega_0 t}\right]\right)$$

$$= \mathrm{Re}\left[\underline{A}\,(i\omega_0)^2 e^{i\omega_0 t}\right] = -\mathrm{Re}\left[\underline{A}\,\omega_0^2 e^{i\omega_0 t}\right]$$

So, putting this into (3.1) gives

$$\cancel{-\mathrm{Re}\left[\underline{A}\,\omega_0^2 e^{i\omega_0 t}\right]} + \cancel{\mu g} + \omega_0^2\left(-\frac{\cancel{\mu g}}{\cancel{\omega_0^2}} + \cancel{\mathrm{Re}\left[\underline{A}e^{i\omega_0 t}\right]}\right)$$

$= 0$ ✓

2. Enough adjustable params:

1 DoF, 2^{nd} order time derivative ⇒ need 2 params

$\underline{A}$ has real and imaginary parts ⇒ have 2 params.

(c) Real Part of $\underline{A}$ (4 points)

For parts (c)-(e), consider initial conditions where, at $t = 0$, the mass is at the equilibrium position $x_0 = x_{\text{eq}} = 0$ and moving with a velocity $v_0 > 0$ *to the right.*

Find the real part $\Re\mathfrak{e}[\underline{A}]$ given the above initial conditions and general solution. Express your answer in terms of only v_0, μ, g, and ω_0. (Note: you may not need all of these.)

$$x_0 = 0 = x(t=0) = -\frac{\mu g}{\omega_0^2} + \mathrm{Re}\left[\underline{A}\, \underbrace{e^{i\omega_0 \cdot 0}}_{1}\right]$$

$$\Rightarrow \boxed{\mathrm{Re}\,\underline{A} = \frac{\mu g}{\omega_0^2}}$$

(d) Imaginary Part of $\underline{A}$ (4 points)

Find the imaginary part $\Im\mathfrak{m}[\underline{A}]$ under the same conditions. Express your answer in terms of only v_0, μ, g, and ω_0. (Note: you may not need all of these.)

$$V_0 = u(t=0) = \frac{dx(t=0)}{dt} = \mathrm{Re}\left[\underline{A} \cdot i\omega_0\, \underbrace{e^{i\omega_0 \cdot 0}}_{1}\right] = -\omega_0\, \mathrm{Im}\,\underline{A}$$

$$\Rightarrow \boxed{\mathrm{Im}\,\underline{A} = \frac{-V_0}{\omega_0}}$$

(e) Maximum distance (4 points) *Challenge Problem!*

What maximum position $x_{\max}$ *will the mass reach before stopping and turning around? Express your answer in terms of only* v_0*,* μ*,* g*, and* ω_0*.* (Note: you may not need all of these.)

$$x_{max} = \max\left(-\frac{\mu g}{\omega_0^2} + \mathrm{Re}\left[\underline{A}e^{i\omega_0 t}\right]\right)$$

$$= -\frac{\mu g}{\omega_0^2} + \max\left(\mathrm{Re}\,\underline{A}e^{i\omega_0 t}\right)$$

$$= -\frac{\mu g}{\omega_0^2} + |\underline{A}| = -\frac{\mu g}{\omega_0^2} + \sqrt{(\mathrm{Re}\,\underline{A})^2 + (\mathrm{Im}\,\underline{A})^2}$$

$$\boxed{x_{max} = -\frac{\mu g}{\omega_0^2} + \sqrt{\left(\frac{\mu g}{\omega_0^2}\right)^2 + \left(\frac{v_0}{\omega_0}\right)^2}}$$

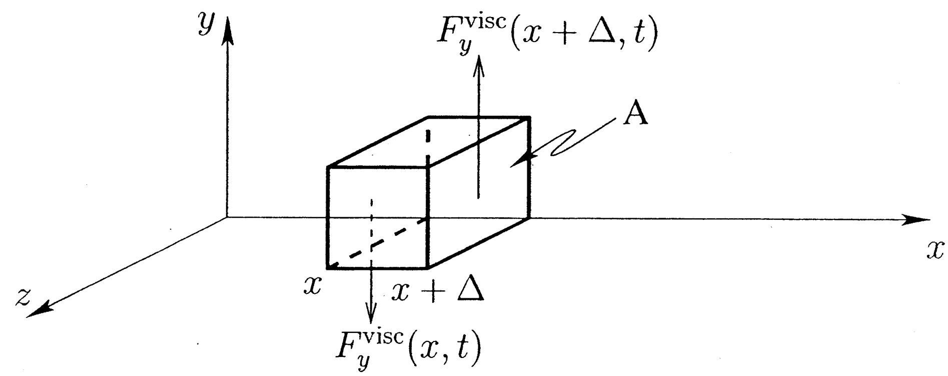


Figure 3: A chunk of fluid experiencing viscous forces

4 Problem 4: Equation of *transverse* motion in a fluid [12 points]

Consider transverse motion along the y direction of a fluid with density ρ_0 and bulk modulus B. For plane waves propagating along x, the constitutive relation for such motion is that the driving force per unit area on a chunk at x due to the adjacent fluid is

$$\frac{F_y^{\text{visc}}(x,t)}{A} = c_v \frac{\partial^2 y(x,t)}{\partial x\, \partial t}, \tag{4.1}$$

with the direction of the force as in Figure 3. Here, c_v is a constant known as the coefficient of viscosity (a constant which is large for "thick" fluids like honey).

Which of the following formulas best represents the equation for transverse motion in a fluid? (Provide your answer in the box on the next page.)

(A) $c_v \dfrac{\partial^3 y(x,t)}{\partial x^2\, \partial t} = \rho_0 \dfrac{\partial^2 y(x,t)}{\partial t^2}$

(B) $c_v \dfrac{\partial^2 y(x,t)}{\partial x\, \partial t} = \rho_0 \dfrac{\partial^2 y(x,t)}{\partial t^2}$

(C) $B \dfrac{\partial^2 y(x,t)}{\partial x^2} = \rho_0 \dfrac{\partial^2 y(x,t)}{\partial t^2}$

(D) $c_v \dfrac{\partial^3 y(x,t)}{\partial x\, \partial t^2} + B \dfrac{\partial^2 y(x,t)}{\partial x^2} = \rho_0 \dfrac{\partial^2 y(x,t)}{\partial t^2}$

(E) $c_v \dfrac{\partial^3 y(x,t)}{\partial x^2\, \partial t} + B \dfrac{\partial^2 y(x,t)}{\partial x^2} = \rho_0 \dfrac{\partial^2 y(x,t)}{\partial t^2}$

$$\frac{\partial}{\partial x}\left(\text{driving force}/A\right) = \rho_0 \frac{\partial^2 y}{\partial t^2}$$

$$c_v \frac{\partial^2 y}{\partial x^2 \partial t} = \rho_0 \frac{\partial^2 y}{\partial t^2}$$

OR

$$\Sigma F_y^{ext} = m a_y$$

$$A c_v \frac{\partial^2 y(x+\Delta,t)}{\partial x \partial t} - A c_v \frac{\partial^2 y(x,t)}{\partial x \partial t} = A \Delta \rho_0 \frac{\partial^2 y}{\partial t}$$

$$c_v \frac{\partial^2 y}{\partial x^2 \partial t} = c_v \frac{\frac{\partial^2 y(x+\Delta,t)}{\partial x \partial t} - \frac{\partial^2 y(x,t)}{\partial x \partial t}}{\Delta} = \rho_0 \frac{\partial^2 y}{\partial t^2}$$

more slowly:

$$\Sigma F_y = 0$$

$$m_{chunk} \frac{\partial^2 y}{\partial t^2} = F_y(x+\Delta, t) - F_y(x,t)$$

$$\rho_0 A \Delta = m_{ch}$$

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = \frac{1}{A} \frac{F_y(x+\Delta,t) - F_y(x,t)}{\Delta}$$

$\Delta \to 0$

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = \frac{1}{A} \frac{\partial F_y(x,t)}{\partial x} = C_V \frac{\partial}{\partial x} \frac{\partial^2 y(x,t)}{\partial x \partial t}$$

const. relation

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = C_V \frac{\partial^3 y(x,t)}{\partial^2 x^2 \partial t} \qquad (A)$$

A

5 Problem 5: Sound Waves versus Waves on a String [12 points]

The following equation is true for sound waves:

$$-\frac{\partial}{\partial x}\left[xB\frac{\partial s}{\partial x} - Bs\right] + \frac{\partial}{\partial t}\left[x\rho_0\frac{\partial s}{\partial t}\right] = 0\,. \tag{5.1}$$

(a) Equation for strings (8 points)

What is the analogous equation for strings?

$s \to y,\ B \to \tau,\ \rho_0 \to \mu:$

$$\boxed{-\frac{\partial}{\partial x}\left(x\tau\frac{\partial y}{\partial x} - \tau y\right) + \frac{\partial}{\partial t}\left(x\mu\frac{\partial y}{\partial t}\right) = 0}$$

(b) Another conservation law (4 points) *Challenge Problem!*

Conservation of what physical quantity does your equation for strings represent?
Hint: The density of the conserved quantity appears inside the $\frac{\partial}{\partial t}[\,]$, and the driving "force" for the conserved quantity appears inside the $\frac{\partial}{\partial x}[\,]$.

"density" $\to \mu = \frac{\text{mass}}{\text{length}}$

Inside $\frac{\partial}{\partial t}[\,]$ is $x\mu\frac{\partial y}{\partial t} = x \cdot p_y = (\vec{r}\times\vec{p})_z$, $\boxed{\text{angular momentum}}$

OR:

Inside $\frac{\partial}{\partial x}[\,]$ is $x\tau\frac{\partial y}{\partial x} - \tau y = xT_y - yT_x = (\vec{r}\times\vec{F})$, torque,

and torque drives $\boxed{\text{angular momentum}}$

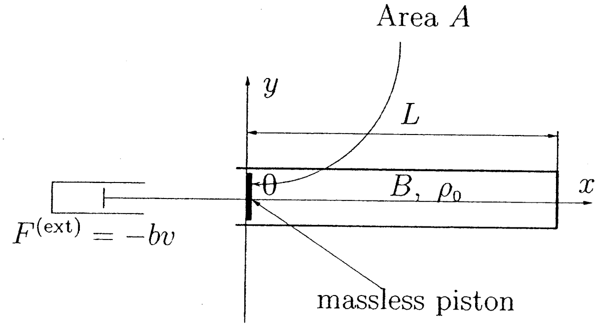


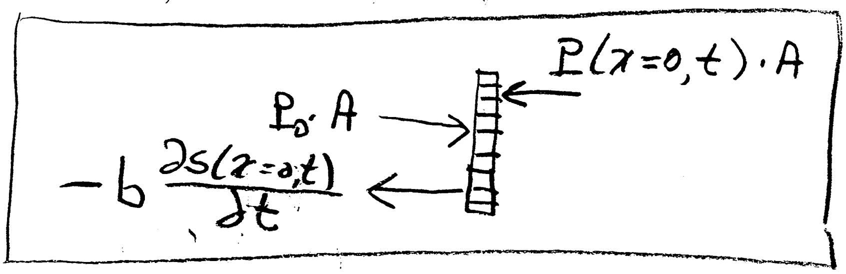
Figure 4: Generalized boundary conditions for a sound tube.

6 Problem 6: Generalized Boundary Conditions for a Sound Tube [16 points]

A tube of length L is filled with air of bulk modulus B and mass density ρ_0 at atmospheric pressure P_0. One end of the tube ($x = L$) is closed. At the other end ($x = 0$), a massless piston of area A can slide freely (without friction) along the tube. The piston is attached to a *shock absorber* which generates a horizontal external force on the piston $\vec{F}^{(\mathrm{ext})} = -b\vec{v}$, where $\vec{v}$ is the velocity of the piston. (See Figure 4.) We denote the sound displacement inside the tube as $s(x, t)$ and the pressure inside the tube as $P(x, t)$.

(a) Air Pressure and Displacement of the Piston (6 points)

Assuming that the piston is moving to the right with velocity $v = \partial s(x = 0)/\partial t$, ***draw a free body diagram for the piston, indicating the directions and the magnitudes of all the forces using only*** $P(x = 0, t)$, $\partial s(x = 0, t)/\partial t$, b, A, ρ_0, L, ***and*** P_0. (Note: you may not need all of these.)



(b) Equation of Motion for the piston (6 points)

Assuming small amplitude waves, so that you can take the location of the piston to be $x \approx 0$, ***write the Equation of Motion for the piston*** in terms of only the constants describing the problem (b, A, B, ρ_0, L, and P_0), and $s(x,t)$ and its derivatives evaluated at $x = 0$. (Note: you may not need all of these.)

$$\sum F_x^{ext} = \cancel{m} \overset{0}{a_x} = 0$$

$$\cancel{P_0} \cdot A - \left(\cancel{P_0} - B \frac{\partial s(x=0,t)}{\partial x}\right) \cdot A - b \frac{\partial s(x,0)}{\partial t} = 0$$

$$p(x=0,t) = p_0 - B \frac{\partial s(x=0,t)}{\partial x}$$

const. relation

$$\Rightarrow \boxed{B \cdot A \cdot \frac{\partial s(x=0,t)}{\partial x} = b \frac{\partial s(x,0)}{\partial t}}$$

(c) Standing wave solutions (4 points) *Challenge Problem!*

In the limit $b \to \infty$, what is the frequency of the lowest frequency standing sound wave in the tube *in terms of* L, ρ_0, *and* B?

As $b \to \infty$, the piston can't move & acts like a closed end. For closed-closed BC, we have from Problem 1, that $\frac{\lambda}{2} = L$. Thus,

$$f = \frac{c}{\lambda} = \frac{c}{2L} = \boxed{\frac{\sqrt{\frac{B}{\rho_0}}}{2L} = f}$$

END OF EXAM