Physics 214 **Fall 2003**

Exercise Problems # 3

1. Off-Axis Diffraction

A single slit of width a is illuminated by an off-axis source. The light from the source has wavelength λ and hits the slit at an angle θ_0. Our goal in this problem is to find out how the single-slit diffraction pattern at a distance screen is changed by having the source off-axis (compared to the "regular" setup where $\theta_0 = 0$).

(a) Consider the slit to be N small slits spaced a distance $d = a/N$ apart, where N is very large. The waves emerging from the various slits do not start in phase, since the light from the source had to travel different distances to reach them. Assuming that the light arrives at the top slit ($n = 0$) with phase ϕ_0, find the phase of the light that arrives at the bottom slit ($n = N - 1$) in terms of θ_0, ϕ_0, a, and λ.

(b) Generalize your result from (a) to find the phase at the n^{th} slit ϕ_n. The n^{th} slit is at a distance $\frac{n}{N}a$ below the top slit. Write your answer in the form $\phi_n = \phi_0 + n\Delta\phi$. Express $\Delta\phi$ in terms of θ_0, ϕ_0, a, l, and N (as needed).

(c) Now use the ϕ_n to find the intensity $I(x)$ at the screen due to the N slits. [*Hint:* Follow the procedure used in section 3.2.1 of the Notes, but this time the ϕ's are not all equal.]

(d) Take the limit $N \to \infty$ to find the intensity at the screen due to the slit of width a. A correct result cannot have any n's or N's in it (why not?). Check your result for the special case $\theta_0 = 0$.

(e) Explain how the intensity pattern differs from the pattern in the "regular" setup. At what angle θ does the "central" maximum appear?

ANSWERS:

(a) $\phi_{N-1} = \phi_0 + ka \sin\theta_0 = \phi_0 + (2\pi a/\lambda)\sin\theta_0$.

(b) $\phi_n = \phi_0 + nk(a/N)\sin\theta_0 = \phi_0 + (2\pi na/N\lambda)\sin\theta_0$. So, $\Delta\phi = (2\pi a/N\lambda)\sin\theta_0$. (This general expression reproduces the result in (a) for very large N: $\phi_{N-1} = \phi_0 + [k(N-1)a/N]\sin\theta_0 \overset{N\to\infty}{\longrightarrow} \phi_0 + ka\sin\theta_0$.)

(c) The arguments of the sines in the N-slit formula have to be shifted by $\Delta\phi$:

$$I(\sin\theta) = I_0 \frac{\sin^2\left[\frac{N}{2}(k\Delta R + \Delta\phi)\right]}{\sin^2\left[\frac{1}{2}(k\Delta R + \Delta\phi)\right]}\,, \qquad \Delta R \equiv (a/N)\sin\theta\,,\ \Delta\phi = (ka/N)\sin\theta_0\,. \tag{1}$$

(d) Taking the limit in a similar way as in class gives:

$$I(\sin\theta) = I_{\max}\frac{\sin^2\left[\frac{ka}{2}(\sin\theta + \sin\theta_0)\right]}{\left[\frac{ka}{2}(\sin\theta + \sin\theta_0)\right]^2}\,. \tag{2}$$

(e) If we plot the intensity I versus θ, the central maximum will appear at $\theta = -\theta_0$.

2. Diffraction at Different Wavelengths:
Young & Freedman, Problem 38-6.

ANSWERS: (a) $2 \cdot 10^{-4}$ m; (b) 2 cm; (c) $2 \cdot 10^{-7}$ m.

3. Multiple Finite Slits:
The graph shows the intensity (in arbitrary units) as a function of position (y) on a screen which is 1.00 m from an aperture illuminated at normal incidence by laser light of wavelength 0.628 μm. The aperture consists of some number of equally spaced, equally wide slits. Determine the number of slits N, the slit width a, and the center-to-center slit separation d.

ANSWERS: The screen intensity $I(\theta)$ for a diffraction pattern from *multiple finite slits* is given by the expression:

$$I(\theta) = \left(I_0 \frac{\sin^2(k\Delta r/2)}{(k\Delta r/2)^2} \right) \left(\frac{\sin^2(Nk\Delta R/2)}{\sin^2(k\Delta R/2)} \right) , \tag{3}$$

where $\Delta r = ka \sin\theta$ and $\Delta R = kd \sin\theta$. Looking at the picture, we therefore determine that:

- Since there are 4 secondary maxima between each two principal maxima, $N = 6$.
- The condition for a principal maximum is $k\Delta R = 2n\pi$ or for $n = 1$, $\sin\theta_1 = \frac{\lambda}{d}$. Then, $d \approx \frac{1\text{m}}{0.015\text{m}} \lambda \approx$ 42 μm.
- The height of the principal maxima is modified by the finite-slit pre-factor in (3). Comparing the heights of the $n = 0$ and $n = 1$ principal maxima on the plot, we obtain $\frac{I_{(n=1)}}{I_{(n=0)}} = \frac{\sin^2(\pi a/d)}{(\pi a/d)^2} = \frac{1.36}{1.8} = 0.76$. The ratio $\frac{a}{d} = \frac{1}{4}$ satisfies this condition reasonably well. Thus, $a \approx d/4 \approx 10.5$ μm.

6. Step in a Box:
A particle of mass m is trapped in an **infinite square well potential** with a **finite step** in the middle. The width of the well is a and the step (of height V_0) is at $x = a/2$. (See Figure 1.)

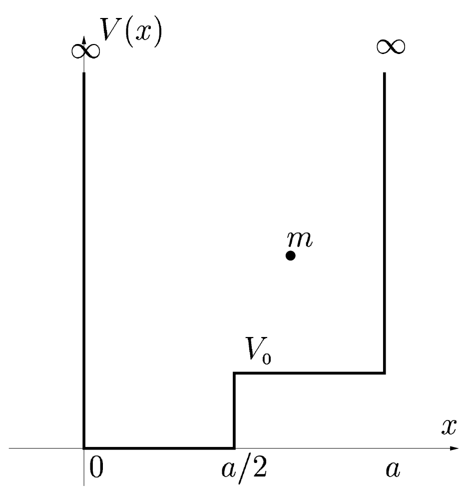

Figure 1: Infinite square well potential.

(a) ***Compute*** the (time-independent) ***wavefunctions*** of the ***two lowest-lying energy eigenstates*** of the particle.

(b) ***Compute*** the ***energies*** of the ***two lowest-lying energy eigenstates*** of the particle.

Hint: Look for solutions of the type $\psi_1(x) = A_1 e^{ikx} + B_1 e^{-ikx}$ in the left region ($x < a/2$) and $\psi_2(x) = A_2 e^{\alpha x} + B_2 e^{-\alpha x}$ in the right region ($x > a/2$). Then, "glue" the wavefunctions smoothly together, i.e., impose the requirements (BCs) that $\psi(x)$ vanish at $x = 0,\ a$, and both $\psi(x)$ and $\dfrac{d\psi}{dx}$ be continuous at $x = a/2$.

ANSWERS: We look for solutions of the type:

$$\psi_1(x) = A_1 e^{ikx} + B_1 e^{-ikx} \ , \tag{4}$$

$$\psi_2(x) = A_2 e^{\alpha x} + B_2 e^{-\alpha x} \ , \tag{5}$$

in the regions 1 and 2 respectively. These functions solve the Schrodinger's equation if $k = \sqrt{2mE}/\hbar$ and $\alpha = \sqrt{2m(V_0 - E)}/\hbar$. Applying the boundary conditions, we get:

$$\psi_1(x = 0) = 0 \Rightarrow B_1 = -A_1 \ , \tag{6}$$

$$\psi_2(x = a) = 0 \Rightarrow B_2 = -A_2 e^{2\alpha a} \ . \tag{7}$$

Using (6), we get $\psi_1(x) = C_1 \sin kx$, where $C_1 = 2iA_1$. Using (7), we get $\psi_2(x) = C_2(e^{\alpha(x-a)} - e^{-\alpha(x-a)})$, where $C_2 = A_2 e^{\alpha a}$. Now, impose BC's at $x = a/2$:

$$\psi_1(x = a/2) = \psi_2(x = a/2) \Rightarrow C_1 \sin(ka/2) = C_2(e^{-\alpha a/2} - e^{\alpha a/2}) \ , \tag{8}$$

$$\psi_1'(x = a/2) = \psi_2'(x = a/2) \Rightarrow kC_1 \cos(ka/2) = C_2 \alpha (e^{-\alpha a/2} + e^{\alpha a/2}) \ . \tag{9}$$

Dividing Eqn. (8) by Eqn. (9), we obtain a transcendental equation

$$\frac{\tan(ka/2)}{k} = -\frac{\tanh(\alpha a/2)}{\alpha} \ , \tag{10}$$

where the "hyperbolic tan" function is defined by

$$\tanh(x) \equiv \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \ . \tag{11}$$

Using our expressions for k and α, we can express everything in the Eqn. (10) in terms of given quantities and the unknown energy of the bound state, E. The two lowest energies will be the *two smallest* solutions to (10), which can be found graphically or numerically. Below, we will assume that we have found E using one of these methods, so that k and α are known.

Once k and α are known, we can substitute them into Eqn. (8) and express C_2 in terms of C_1. So, we get:

$$\psi_1(x) = C_1 \sin(kx) \tag{12}$$

$$\psi_2(x) = C_1 \sin(ka/2)\, \frac{e^{\alpha(x-a)} - e^{-\alpha(x-a)}}{e^{-\alpha a/2} - e^{\alpha a/2}} \ . \tag{13}$$

This gives the wavefunction, up to a single "normalization" constant (determining the vertical scale) C_1. This constant is determined by the "normalization condition",

$$\int_0^a dx\, |\psi(x)|^2 = 1 \ . \tag{14}$$

(This just says that the *total* probability of finding the particle somewhere in the well is equal to 1.) Given this condition, it is straightforward (but tedious) to calculate C_1; you are *not* required to normalize your wavefunctions correctly in this course, so you may leave C_1 as an arbitrary constant in the answers.

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7. Finite Well

Now consider a particle of mass m confined in an **finite square well potential** of height V_0 and width a. (See Figure 2.)

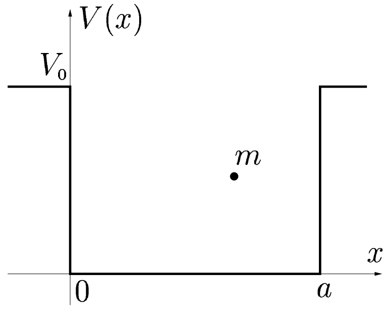

Figure 2: Finite square well potential.

(a) ***Compute*** the (time-independent) ***wavefunctions*** of the ***two lowest-energy states*** of the particle.

Hint: Use the same strategy as in the previous problem. In this case the smoothness requirements (BCs) are that both $\psi(x)$ and $\frac{d\psi}{dx}$ be continuous at $x = 0,\ a$.

(b) ***Write down the equations*** that the ***energies*** of the ***two lowest-energy states must satisfy.***

Hint: Some of the boundary conditions above will impose constraints on the wavevectors $k = \sqrt{2mE}/\hbar$ and $k' = \sqrt{2m(E - V_0)}/\hbar$, and thus on the energy E. As a result the energy can only take certain discrete values, solutions to transcendental equations (e.g., $x = \tan x$ is a *transcendental equation*). Such equations can only be solved numerically (or graphically). In this problem you are not required to solve the equations; it is enough just to write them down.

ANSWERS: We look for solutions of the type:

$$\psi_1(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x}\ , \tag{15}$$
$$\psi_2(x) = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x}\ , \tag{16}$$
$$\psi_3(x) = A_3 e^{ik_3 (x-a)} + B_3 e^{-ik_3 (x-a)}\ . \tag{17}$$

in the regions 1, 2, and 3 respectively, with $k_1 = k_3 = i\sqrt{2m(V_0 - E)}/\hbar = i\kappa$ and $k_2 = \sqrt{2mE}/\hbar$. Note that since we are looking for the lowest-energy bound states, we take $E < V_0$ and the wavevectors in the regions 1 and 3 are *imaginary.* The exponents in (15) and (17) are therefore *real* and in order to ensure that the wavefunction vanishes at infinity, we must set $A_1 = B_3 = 0$. (The right-moving waves that penetrate region 3 and the left-moving waves that penetrate region 1 decay exponentially; there is no right-moving waves in region 1 and no left-moving waves in region 3.)

Applying the boundary conditions at $x = 0$ and $x = a$, we get:

$$\psi_1(x=0) = \psi_2(x=0) \Rightarrow B_1 = A_2 + B_2\ , \tag{18}$$
$$\psi_1'(x=0) = \psi_2'(x=0) \Rightarrow \kappa B_1 = ik_2(A_2 - B_2)\ , \tag{19}$$
$$\psi_1(x=a) = \psi_2(x=a) \Rightarrow A_3 = A_2 e^{ik_2 a} + B_2 e^{-ik_2 a}\ , \tag{20}$$
$$\psi_1'(x=0) = \psi_2'(x=0) \Rightarrow -\kappa A_3 = ik_2(A_2 e^{ik_2 a} - B_2 e^{-ik_2 a})\ . \tag{21}$$

We can solve Eqs. (18–21) in terms of a single constant A (to be fixed by normalization):

$$B_1 = 2ik_2 A\ , \tag{22}$$
$$A_2 = A(ik_2 + \kappa)\ , \tag{23}$$
$$B_2 = A(ik_2 - \kappa)\ , \tag{24}$$
$$A_3 = 2iA(k_2 \cos(k_2 a) + \kappa \sin(k_2 a))\ . \tag{25}$$

The wavefunctions are of the form (15–17) with B_1, A_2, B_2, and A_3 given by (22–25) and k_2, κ can be obtained once we know the value of the energy (recall that $k_2 = \sqrt{2mE}/\hbar$ and $\kappa = \sqrt{2m(V_0 - E)}/\hbar$).

The energy can only have *discrete* values, solutions to the transcendental equation:

$$\frac{2k_2\kappa}{k_2^2 - \kappa^2} = \tan(k_2 a) \ . \tag{26}$$

The two lowest energies will be the *two smallest* solutions to (26), and the two lowest states can be obtained by substituting those values for the energy into (22–25) and (15–17).