④ 4.1 Change in Medium.

(a) $c = \sqrt{\frac{\tau}{\mu}} \Rightarrow c' = \sqrt{\frac{\tau}{\mu'}} = \sqrt{\frac{\tau}{9\mu}} = \frac{1}{3}\sqrt{\frac{\tau}{\mu}} = \frac{1}{3} c \approx 33 \frac{m}{sec}$.

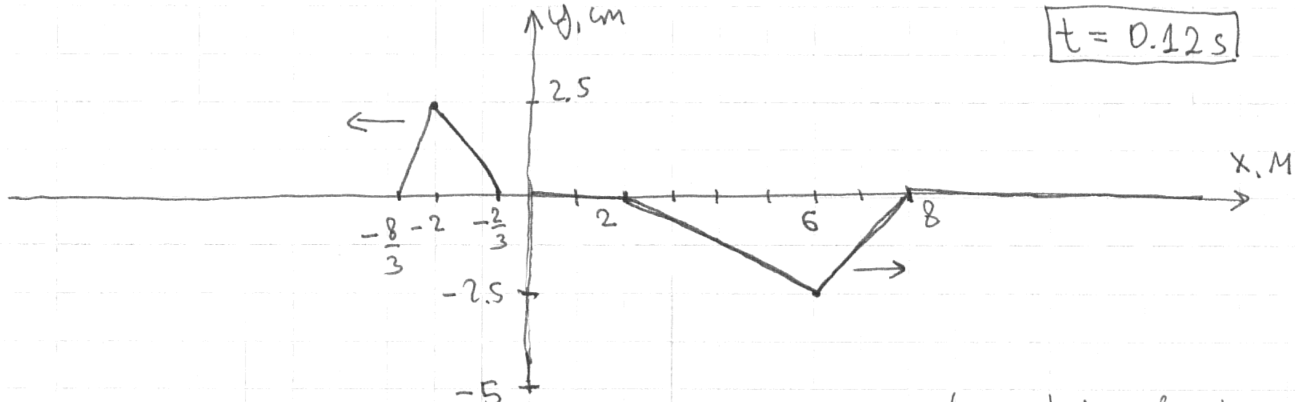
Reflection & transmission coefficients:

$$R = \frac{\mu c - \mu' c'}{\mu c + \mu' c'} = \frac{\mu c - 9\mu \cdot \frac{1}{3} c}{\mu c + 9\mu \cdot \frac{1}{3} c} = -\frac{1}{2}.$$

$$T = \frac{2\mu c}{\mu c + \mu' c'} = +\frac{1}{2}$$

(we use $z = \mu c$ + standard formulas for R & T.)

Transmitted pulse is shrunk horizontally by $c/c' = 3$.




(see below for t=0.06s!)

(b) energies: $pe = \frac{1}{2}\tau \times \left(\frac{\partial y}{\partial x}\right)^2$, total $e = 2 \cdot pe$ in a pulse.

Incoming: (at t=0)

$4 \le x \le 6$, $pe = \frac{1}{2}\tau \times \left(\frac{5\,cm}{2\,m}\right)^2 = \frac{25}{8} \times 10^{-4} \times \tau$ $[\frac{J}{m}]$

$6 \le x \le 10$, $pe = \frac{1}{2} \cdot \tau \times \left(\frac{-5\,cm}{4\,m}\right)^2 = \frac{25}{32} \times 10^{-4} \times \tau$

total:

$$\text{Energy} = 2 \times \left(\frac{25}{8} \times 2 + \frac{25}{32} \times 4\right) \times 10^{-4}\,\tau$$

$$= \frac{75}{4} \times 10^{-4}\,\tau.$$

Reflected: $2 \le x \le 6$, $pe = \frac{1}{2}\tau\cdot\left(\frac{-2.5\,cm}{4m}\right)^2 = \frac{25}{128}\times 10^{-4}\times\tau$

(at t=0.12sec)

$6 \le x \le 8$, $pe = \frac{1}{2}\tau\left(\frac{+2.5\,cm}{2m}\right)^2 = \frac{25}{32}\times 10^{-4}\times\tau$

Total Energy $= 2\times\left(\frac{25}{128}\times 4 + \frac{25}{32}\times 2\right)\times 10^{-4}\times\tau$

$= \frac{75}{16}\times 10^{-4}\times\tau$

Transmitted: $-\frac{8}{3} \le x \le -2$, $pe = \frac{1}{2}\cdot\tau\cdot\left(\frac{2.5\,cm}{2/3\,m}\right)^2$

(at t=0.12sec)

$= \frac{9\cdot 25}{32}\times 10^{-4}\times\tau$

$-2 \le x \le -2/3$: $pe = \frac{1}{2}\cdot\tau\cdot\left(\frac{-2.5\,cm}{4/3\,m}\right)^2$

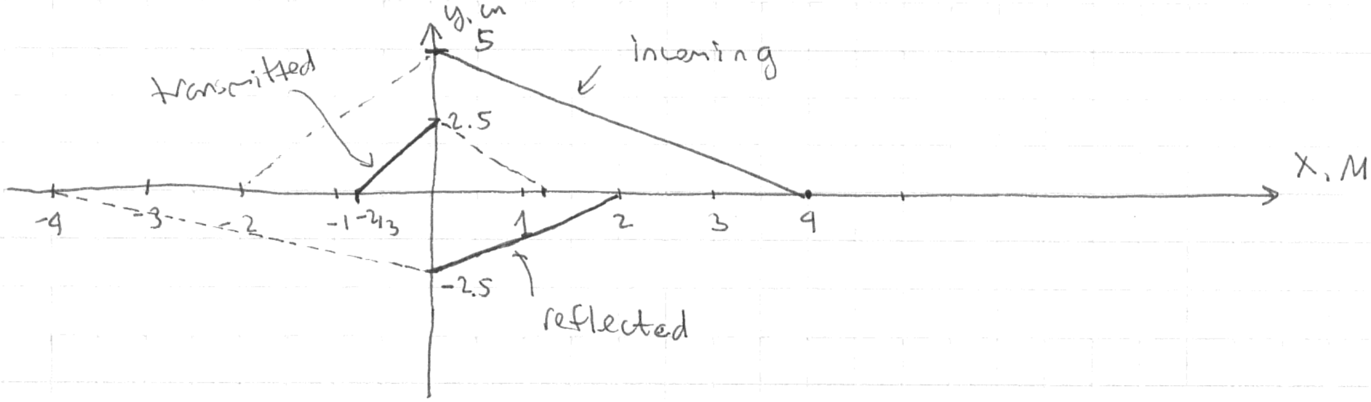
$= \frac{9\cdot 25}{128}\times 10^{-4}\times\tau$

Total energy $= 2\times\left(\frac{9\cdot 25}{32}\times\frac{2}{3} + \frac{9\cdot 25}{128}\times\frac{4}{3}\right)\times 10^{-4}\times\tau$

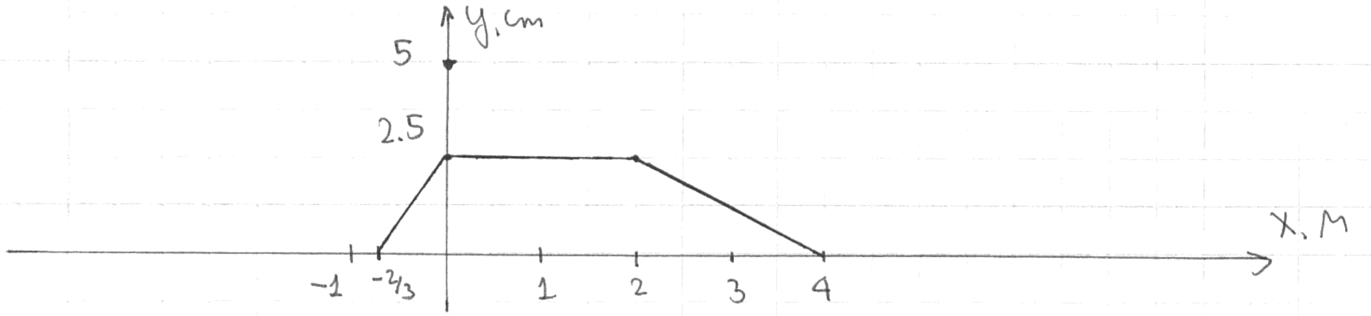
$= 3\cdot\frac{75}{16}\times 10^{-4}\,\tau$

$\underset{\text{transmitted}}{3\cdot\frac{75}{16}} + \underset{\text{reflected}}{\frac{75}{16}} = \underset{\text{incoming}}{\frac{75}{4}}$ $\Rightarrow$ energy is conserved!

(a) Plot for $t = 0.06$ s




Add them up graphically:

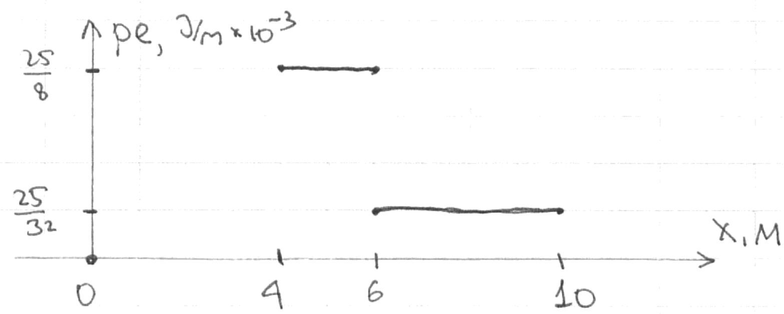



## 4.2 Reflection

(a) using calculations in 4.1 b):

$4 \le x \le 6$, $pe = \frac{25}{8} \times 10^{-3}$ $\frac{J}{m}$;

$6 \le x \le 10$, $pe = \frac{25}{32} \times 10^{-3}$ $\frac{J}{m}$




$ke(x) = pe(x)$; $e(x) = 2 \cdot pe(x)$

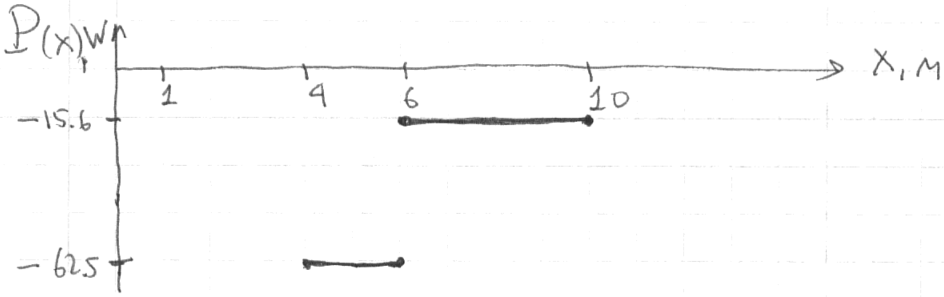
Power: $P = -\tau \frac{\partial y}{\partial x} \frac{\partial y}{\partial t}$

Pulse: $y = f(x + ct)$ (leftmoving!)

$\Rightarrow \frac{\partial y}{\partial t} = cf'$; $P = -\tau \cdot c \cdot (f')^2$

$4 \leq x \leq 6$, $P = -\frac{250}{4} = -62.5$ W

$6 \leq x \leq 10$, $P = -\frac{250}{16} \approx -15.6$ W




(b) $P(x) = -c\, e(x)$ : energy is flowing with speed $c$, in the negative $x$ direction!

(c) Fixed end $\Rightarrow \frac{\partial y}{\partial t} = 0 \Rightarrow P = 0$

Reflected pulse has exactly the same energy as the incoming pulse (it's just inverted) $\Rightarrow$ no energy flows into $x < 0$ region.

(d) Free end $\Rightarrow \frac{\partial y}{\partial x} = 0 \Rightarrow P = 0$

Same reasoning as in (c).

## 4.3 Perfect Absorber

(a) At $x = 0$, the boundary condition is

$$\tau \frac{\partial y}{\partial x}(x=0,t) = b \frac{\partial y}{\partial t}(x=0,t)$$

(see pr. 5 on problem set #7).

Using this,
$$P = -\tau \frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = -\frac{\tau^2}{b}\left(\frac{\partial y}{\partial x}\right)^2$$

But $b = \frac{\tau}{c}$, so $$P = -\tau c \left(\frac{\partial y}{\partial x}\right)^2.$$

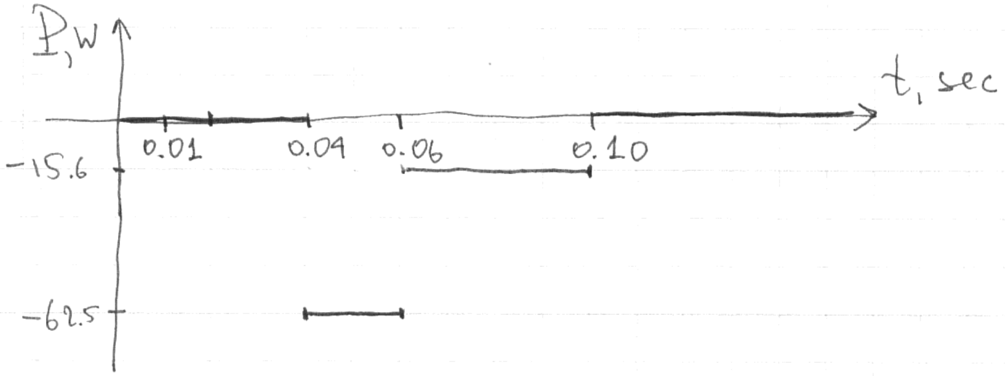
There is no reflected pulse $\Rightarrow$ $\frac{\partial y}{\partial x}$ computed from incoming alone.

$0.04 \text{ sec} \le t \le 0.06 \text{ sec}$: $\frac{\partial y}{\partial x} = \frac{5 \text{ cm}}{2 \text{ m}} = 2.5 \times 10^{-2}$

$$P = -62.5 \text{ W}$$

$0.06 \text{ sec} \le t \le 0.10 \text{ sec}$: $\frac{\partial y}{\partial x} = \frac{-5 \text{ cm}}{4 \text{ m}} = -1.25 \times 10^{-2}$,

$$P \approx -15.6 \text{ W}$$




(b) $P_d = F_y v_y = -b\left(\frac{\partial y}{\partial t}\right)^2$; but $\frac{\partial y}{\partial t} = \frac{\partial y}{\partial x}\cdot c$, so that

$$P_d = -b \cdot c^2 \cdot \left(\frac{\partial y}{\partial x}\right)^2$$

But $b = \frac{T}{c}$ $\Rightarrow$ $P_d = -T \cdot c \cdot \left(\frac{\partial y}{\partial x}\right)^2$

So $P_d(t) = P(t)$.

Since there is no reflected pulse, all the energy of the incoming pulse has to be dissipated by the dashpot. This is exactly what $P_d(t) = P(t)$ means.

(5) (a) $\psi(-a) = \psi(a) = 0 \Rightarrow$ analogous to standing waves on a string with 2 fixed ends!

More formally: $\psi(x) = A\cos kx + B\sin kx, \quad -a < x < a; \ \psi = 0, |x| > a$

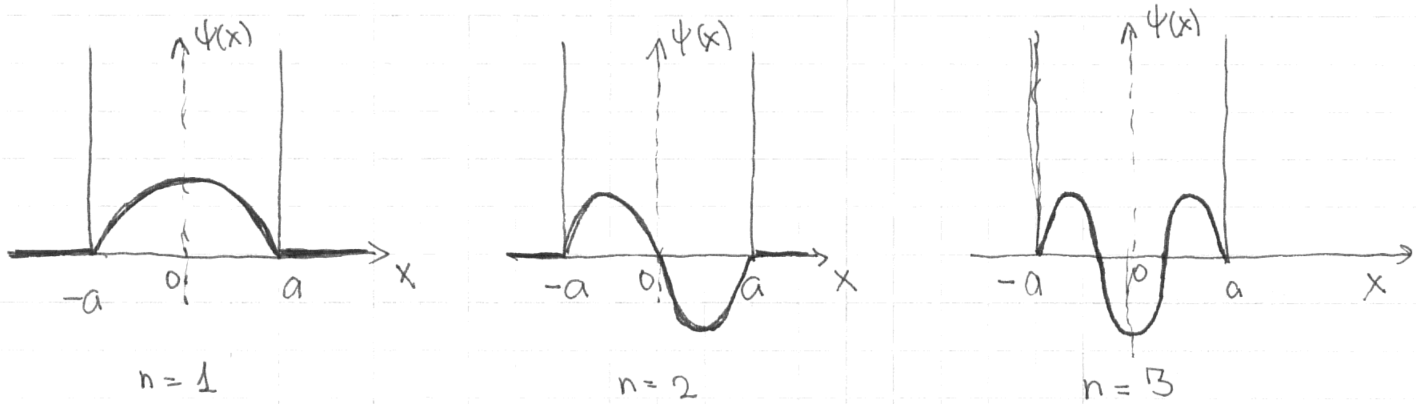
$$\psi(a) = 0 \Rightarrow \psi(x) = A\sin k(x-a)$$

$$\psi(-a) = 0 \Rightarrow A\sin k(-2a) = 0 \Rightarrow 2ak = \pi n, \ n = 0, 1, 2, \ldots$$

$$\Rightarrow k = \frac{\pi n}{2a}, \quad \text{and} \quad \boxed{\psi(x) = A\sin\frac{\pi(x-a)}{2a}n}.$$

$n = 0 \Rightarrow \psi \equiv 0$, not a "state" (probability = 0 everywhere!)

$\Rightarrow$ plot $n = 1, 2, 3$




(Note: we do not worry about precisely determining the constant "A"!)

(b) $E = \frac{p^2}{2m}$; De Broglie hypothesis $\Rightarrow p = \hbar k$

$$E_n = \frac{(\hbar k_n)^2}{2m} = \frac{\hbar^2}{2m}k_n^2$$

But $k_n = \frac{\pi n}{2a} \Rightarrow E_n = \frac{\hbar^2\pi^2}{8ma^2}n^2.$

Using this, find

$$\boxed{E_1 = \frac{\hbar^2\pi^2}{8ma^2}, \quad E_2 = \frac{\hbar^2\pi^2}{2ma^2}, \quad E_3 = \frac{9\hbar^2\pi^2}{8ma^2}}$$

(c) We have an equation for the ground state energy:

$$E_1 = \frac{\hbar^2\pi^2}{8MA^2}$$

where $M$ = particle mass, $A = \frac{1}{2}$ width of the box. So:

(A): $M = m$, $A = a$, $E_1 = \frac{\hbar^2\pi^2}{8Ma^2}$

(B): $M = \frac{m}{2}$, $A = 2a$, $E_1 = \frac{\hbar^2\pi^2}{8\left(\frac{m}{2}\right)\times 4a^2} = \frac{\hbar^2\pi^2}{16ma^2}$

(C): $M = 2m$, $A = a/2$, $E_1 = \frac{\hbar^2\pi^2}{8\cdot 2m\cdot(a/2)^2} = \frac{\hbar^2\pi^2}{4ma^2}$

(D): $M = 2m$, $A = a$, $E_1 = \frac{\hbar^2\pi^2}{8\cdot 2m\cdot a^2} = \frac{\hbar^2\pi^2}{16ma^2}$

$\Rightarrow$ $\boxed{\text{(B) and (D)}}$ have the same ground state energy.

## 8 Heisenberg uncertainty principle & Hydrogen.

(a)


Electron wavefunction is "spread out" over distances $\sim r$, meaning that there is some probability, at any given time, to find an electron anywhere within the "cloud" of radius $r$, centered at the nucleus. So, the <u>exact</u> position of the electron is unknown, $\Delta x \sim r$ is the uncertainty in the position. H.u.p. says

$$\Delta x \cdot \Delta p \gtrsim \hbar$$

for <u>any</u> system; for a microscopic system like H atom, this inequality is (nearly) saturated,

$$\Delta x \cdot \Delta p \approx \hbar$$

and so

$$\Delta p \approx \frac{\hbar}{\Delta x} \approx \frac{\hbar}{r}.$$

(b)

$$E = K + P, \qquad K = \frac{(\Delta p)^2}{2m} \approx \frac{\hbar^2}{2mr^2};$$

$$P = eV = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

So, total energy is

$$\boxed{E = \frac{\hbar^2}{2m}\frac{1}{r^2} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}.}$$

(c) $$\frac{dE}{dr} = \frac{\hbar^2}{2m}\left(-\frac{2}{r^3}\right) + \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}$$

To find $r_0$: $\left.\frac{dE}{dr}\right|_{r_0} = 0$, $$-\frac{\hbar^2}{m}\frac{1}{r_0^3} + \frac{e^2}{4\pi\epsilon_0}\frac{1}{r_0^2} = 0$$

$$\Rightarrow \boxed{r_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}}$$

$$E_{min} = E(r_0) = \frac{\hbar^2}{2m}\left(\frac{me^2}{4\pi\epsilon_0\hbar^2}\right)^2 - \frac{e^2}{4\pi\epsilon_0}\left(\frac{me^2}{4\pi\epsilon_0\hbar^2}\right) \Rightarrow$$

$$\boxed{E_{min} = -\frac{1}{2}\frac{me^4}{(4\pi\epsilon_0)^2\hbar^2}}$$