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Homework #10 Solutions

1

$$y(x,t) = A\sin(kx)\cos(\omega t)$$

$$\frac{\partial y(x,t)}{\partial t} = -A\omega\sin(kx)\sin(\omega t)$$

$$\frac{\partial y(x,t)}{\partial x} = Ak\cos(kx)\cos(\omega t)$$

Trig identities used

$\sin(A+B) = \sin A\cos B + \sin B\cos A$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\sin^2\theta + \cos^2\theta = 1$, $\cos^2\theta - \sin^2\theta = (\cos\theta - \sin\theta) \times (\cos\theta + \sin\theta)$

a) $ke(x,t) = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 = \frac{A^2\omega^2\mu}{2}\sin^2(kx)\sin^2(\omega t)$

$pe(x,t) = \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 = \frac{A^2k^2\tau}{2}\cos^2(kx)\cos^2(\omega t)$

$e(x,t) = ke(x,t) + pe(x,t) = \frac{A^2\omega^2\mu}{2}\left[\sin^2(kx)\sin^2(\omega t) + \cos^2(kx)\cos^2(\omega t)\right]$

where $v = \frac{\omega}{k} = \sqrt{\frac{\tau}{\mu}} \Rightarrow \omega^2\mu = \tau k^2$

Therefore $e(x,t)$ varies with time. To check if energy is conserved we must calculate the total energy.

$$E = \int_0^L e(x,t)\,dx = \frac{A^2\omega^2\mu}{2}\int_0^L\left[\sin^2(kx)\sin^2(\omega t) + \cos^2(kx)\cos^2(\omega t)\right]dx$$

$$= \frac{A^2\omega^2\mu L}{4}\left[\sin^2(\omega t) + \cos^2(\omega t)\right], \quad \text{b/c } \int_0^L \sin^2(kx)\,dx = \int_0^L \cos^2(kx)\,dx = \frac{L}{2}$$

$$= \frac{A^2\omega^2\mu L}{4}$$ a constant. Therefore energy is conserved.

b) $P(x,t) = -\tau\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = A^2\omega k\tau\sin(kx)\cos(kx)\sin(\omega t)\cos(\omega t)$

$$= \frac{A^2\omega k\tau}{4}\sin(2kx)\sin(2\omega t)$$

c) $\frac{\partial e}{\partial t} = A^2\omega^3\mu\left[\sin^2(kx)\sin(\omega t)\cos(\omega t) - \cos^2(kx)\sin(\omega t)\cos(\omega t)\right]$

$$= -\frac{A^2\omega^3\mu}{2}\left[\cos(2kx)\sin(2\omega t)\right]$$

|| b/c $\omega^2\mu = k^2\tau$

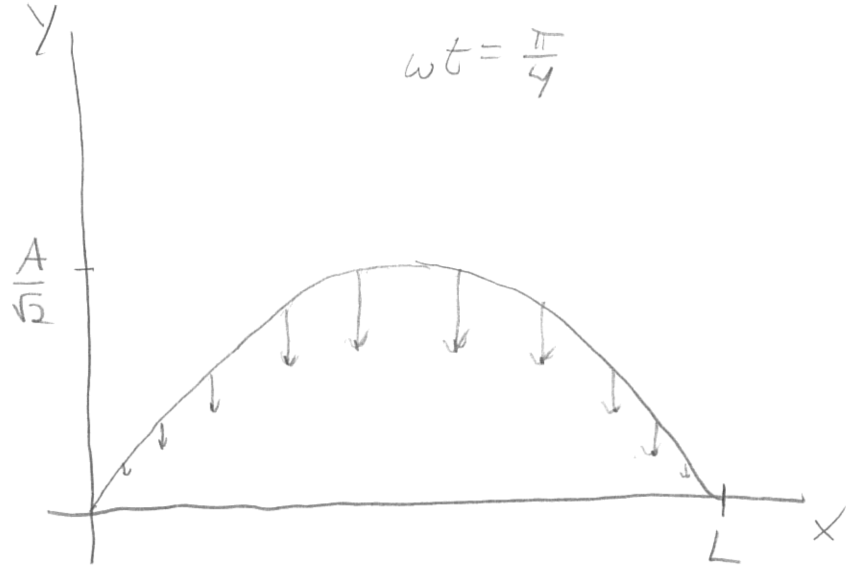
$$-\frac{\partial P}{\partial x} = -\frac{A^2\omega k^2\tau}{2}\left[\cos(2kx)\sin(2\omega t)\right]$$

d) lowest fundamental frequency

$\Rightarrow k = \frac{\pi}{L}$

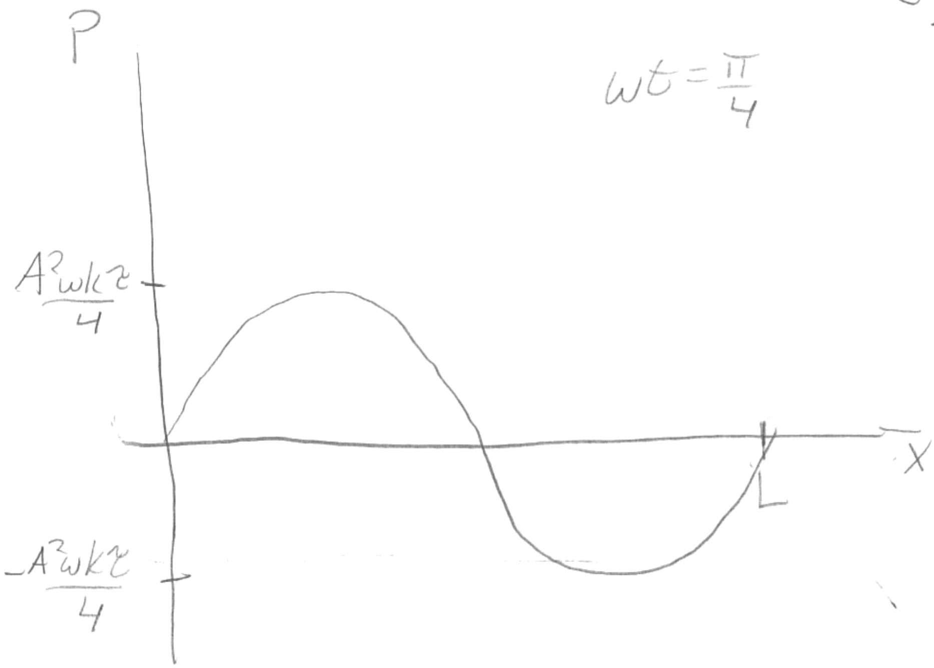
$$\Rightarrow y(x, t_0) = A\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi}{4}\right)$$

$$= \frac{A}{\sqrt{2}}\sin\left(\frac{\pi x}{L}\right)$$



$$P(x,t) = \frac{A^2 \omega k \tau}{4}\sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{\pi}{2}\right)$$

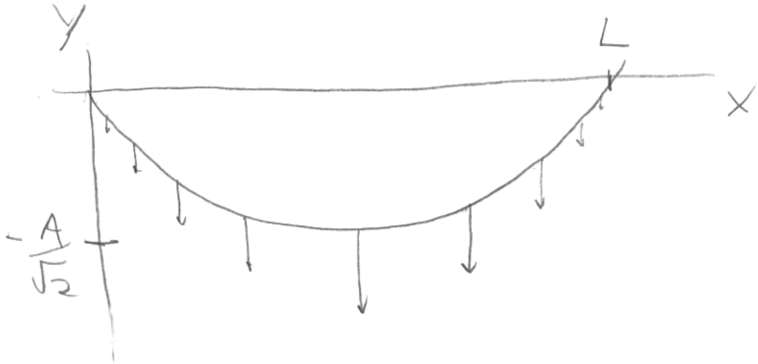
$$= \frac{A^2 \omega k \tau}{4}\sin\left(\frac{2\pi}{L}x\right)$$



e) Energy is moving to the right for $0 < x < \frac{L}{2}$ and energy is moving to the left for $\frac{L}{2} < x < L$

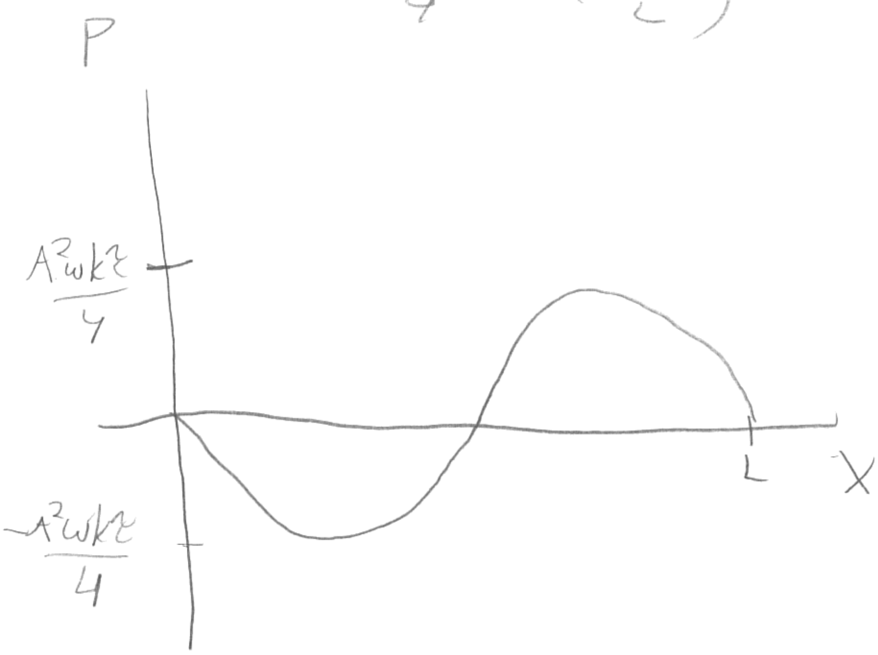
f) $y(x, t_0) = A\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{3\pi}{4}\right)$

$$= -\frac{A}{\sqrt{2}}\sin\left(\frac{\pi x}{L}\right)$$



$$P(x, t_0) = \frac{A^2\omega k\tau}{4}\sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{3\pi}{2}\right)$$

$$= -\frac{A^2\omega k\tau}{4}\sin\left(\frac{2\pi x}{L}\right)$$



$0 < x < \frac{L}{2}$ Energy is moving to the left

$\frac{L}{2} < x < L$ Energy is moving to the right

2) a) Electric field energy

$\frac{1}{2}\varepsilon|E|^2$

$= \frac{1}{2}\varepsilon\left|Re\left(E(x)e^{i(kx-\omega t)}\right)\hat{y}\right|^2$

where $\underline{E}(x) = E(x)e^{ikx}$

$= \frac{\varepsilon}{2}E(x)^2\cos^2(kx-\omega t)$

Magnetic field energy

$\frac{1}{2\mu}|B|^2$

$= \frac{1}{2\mu}\left|\frac{1}{c}Re\left(E(x)e^{i(kx-\omega t)}\right)\hat{z}\right|^2$

$= \frac{1}{2\mu c^2}E(x)^2\cos^2(kx-\omega t)$

$= \frac{\varepsilon}{2}E(x)^2\cos^2(kx-\omega t)$

b) Total energy density

$u_{em} = \varepsilon E(x)^2\cos^2(kx-\omega t)$

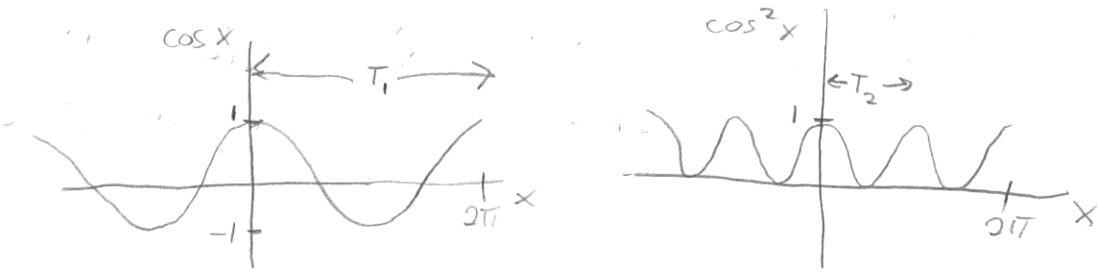
c) $\vec{S}(x,t) = \frac{1}{\mu}\vec{E}\times\vec{B} = \frac{1}{\mu}\left(E(x)\cos(kx-\omega t)\hat{y}\right)\times\left(\frac{E(x)}{c}\cos(kx-\omega t)\hat{z}\right)$

$= \frac{1}{\mu c}E(x)^2\cos^2(kx-\omega t)\hat{x}$

energy is moving to the right.

$S(x,t) = |\vec{S}(x,t)| = \frac{1}{\mu c}E(x)^2\cos^2(kx-\omega t)$

d)



Claim: $S(x,t) = S(x, t+\frac{\pi}{\omega})$

$S(x,t+\frac{\pi}{\omega}) = \frac{1}{\mu c}E(x)^2\cos^2(kx-\omega t-\pi)$

$= \frac{1}{\mu c}E(x)^2\left[\cos(kx-\omega t)\cancel{\cos(\pi)}^{-1} + \sin(kx-\omega t)\cancel{\sin(\pi)}^{0}\right]^2$

$= \frac{1}{\mu c}E(x)^2\cos^2(kx-\omega t) = S(x,t)$

Therefore $S(x,t)$ is periodic in time. By looking at the above graphs we can see that the period of $S(x,t)$ is $\frac{\pi}{\omega}$ while the period of $\vec{E}$ and $\vec{B}$ is $\frac{2\pi}{\omega} \Rightarrow$ period of $S(x,t) = \frac{T}{2}$

e) $T = \frac{\lambda}{c}$ $T_{blue} = \frac{400 \times 10^{-9} m}{3 \times 10^{8} \frac{m}{s}} = 1.33 \times 10^{-15} s$

$$T_{red} = \frac{700 \times 10^{-9} m}{3 \times 10^{8} \frac{m}{s}} = 2.33 \times 10^{-15} s$$

f) S is periodic in time with a period on the order of 10^{-15} s. Therefore, it would be very difficult to make precise measurements of S because it is changing so fast. We define the intensity of light to be the average value of S because this value can be measured more easily.

h) $I = \langle S(x,t) \rangle = \langle \frac{1}{\mu c} E(x)^2 \cos^2(kx - \omega t) \rangle$

$$= \frac{1}{2\mu c}(E(x))^2 = \frac{1}{2\mu c}|E(x)|^2$$

$$\Rightarrow \alpha = \frac{1}{2\mu c} = \frac{1 \sqrt{\varepsilon \mu}}{2\mu} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}$$

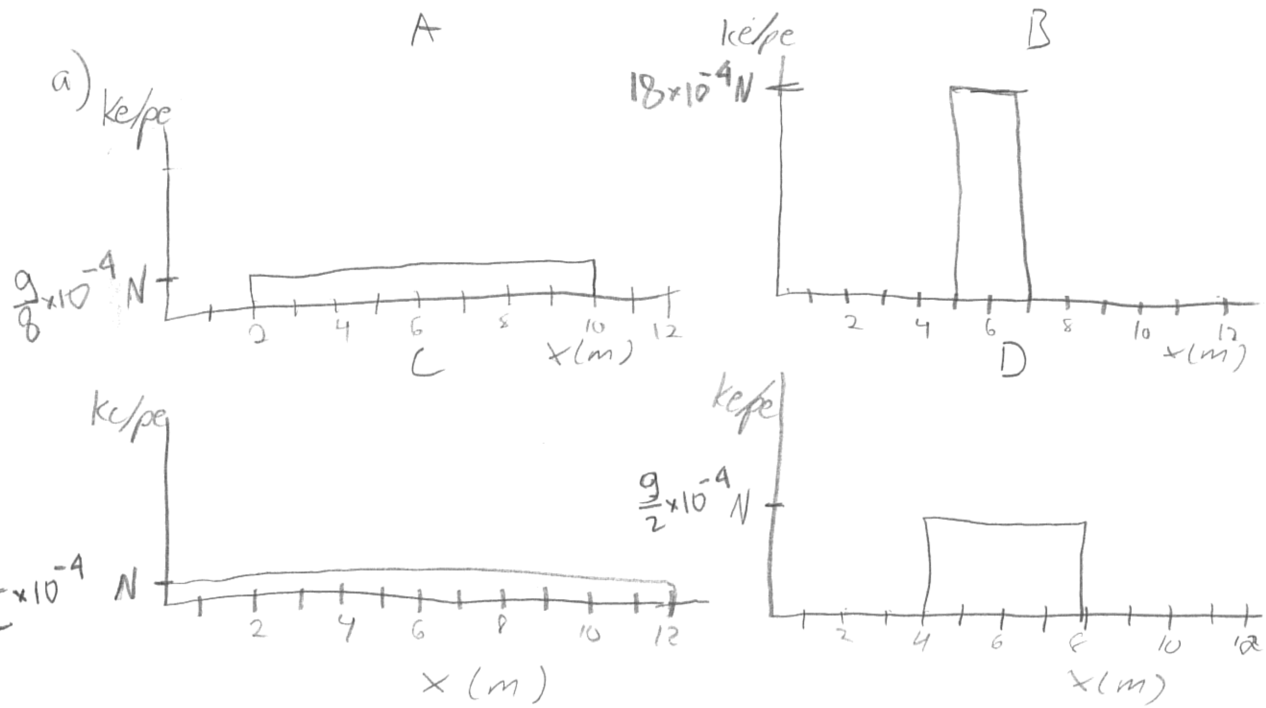
(3)

$$ke(x,t) = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\mu\left(\frac{\partial y}{\partial x}\frac{\partial x}{\partial t}\right)^2 = \frac{\mu}{2}\left(-c\frac{\partial y}{\partial x}\right)^2 = \frac{c^2\mu}{2}\left(\frac{\partial y}{\partial x}\right)^2$$

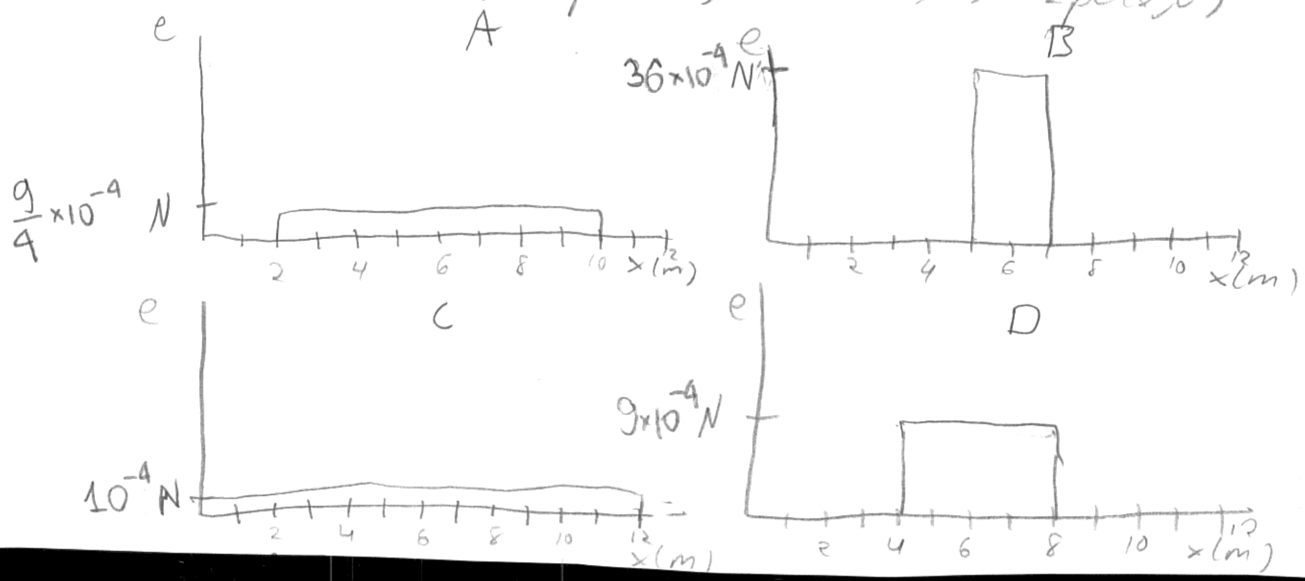
$$= \frac{\tau}{2}\left(\frac{\partial y}{\partial x}\right)^2 = pe(x,t)$$

Note: The above derivation only works for single pulses. $ke(x,t) \neq pe(x,t)$ when there are overlapping pulses moving in different directions or at different speeds.

a)



b) $e(x,t) = ke(x,t) + pe(x,t) = 2ke(x,t) = 2pe(x,t)$



(c) $E_A = \left(\frac{9}{4} \times 10^{-4} N\right) \times 8 m = 18 \times 10^{-4} J$;

$E_B = (36 \times 10^{-4} N) \times 2 m = 72 \times 10^{-4} J$;

$E_c = (10^{-4} N) \times 12 m = 12 \times 10^{-4} J$;

$E_D = (9 \times 10^{-4} N) \times 4 m = 36 \times 10^{-4} J$

"<u>B</u>" is the pulse with the highest total energy.

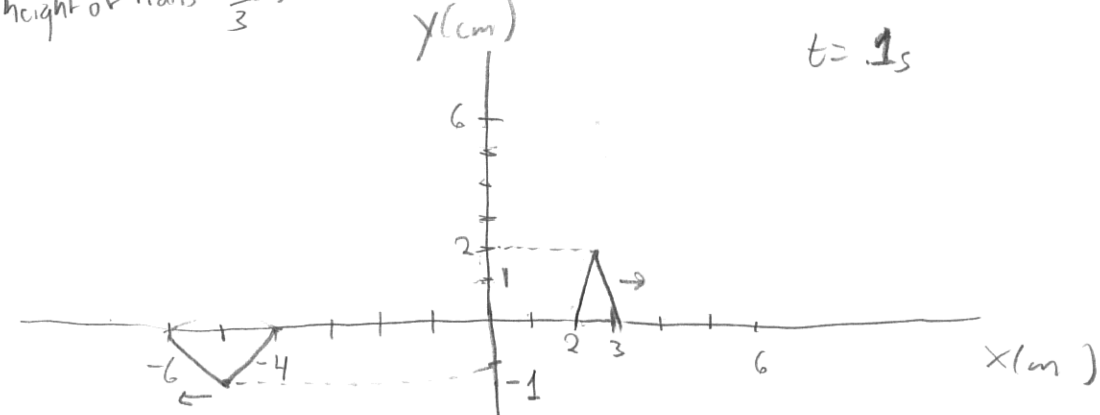
4. $T = \frac{2Z_1}{Z_2+Z_1}$, $R = \frac{Z_1 - Z_2}{Z_2 + Z_1}$, $Z_1 = \frac{\tau}{100\frac{m}{s}}$ $Z_2 = \frac{\tau}{50\frac{m}{s}}$

$\Rightarrow Z_2 = 2Z_1$

$= \frac{2Z_1}{3Z_1} = \frac{2}{3}$ $\qquad = \frac{-Z_1}{3Z_1} = \frac{-1}{3} \Rightarrow$ height of reflected $= \frac{1}{3}(3) = 1cm$ + inverted!

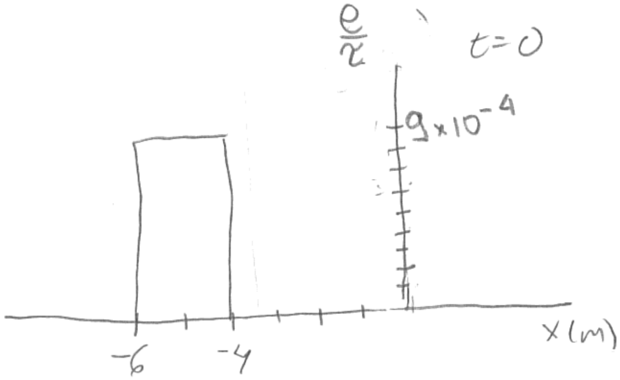
$\Rightarrow$ height of trans $= \frac{2}{3}(3) = 2cm$

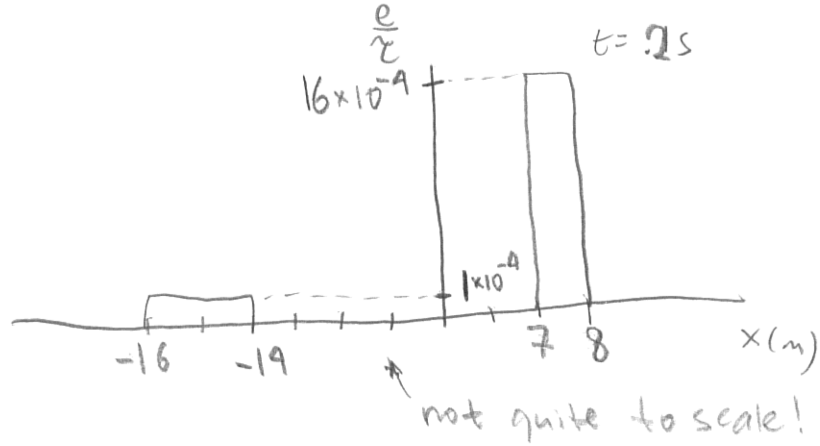
a)



b) For a single pulse (one that does not overlap with others)

$e(x,t) = 2p_e(x,t) = \tau\left(\frac{\partial y}{\partial x}\right)^2$





c) $E = \int_{-\infty}^{\infty} e(x,t)\,dx$

incoming pulse: $E_i = (9\tau)(2m) \times 10^{-4} = 18 \times 10^{-4}\ \tau \cdot m$

reflected pulse: $E_r = (1\tau)(2m) \times 10^{-4} = 2 \times 10^{-4}\ \tau \cdot m$

transmitted pulse: $E_t = (16\tau)(1m) \times 10^{-4} = 16 \times 10^{-4}\ \tau \cdot m$

$E(t=0s) = (18m)\tau \times 10^{-4}$

$E(t=.1s) = E_r + E_t = 2\times10^{-4} + 16\times10^{-4} = (18m)\cdot\tau\times10^{-4}$!

$\Rightarrow E(t=0s) = E(t=.1s) \Rightarrow$ Energy is conserved.