Cornell University

Department of Physics

Phys 214 November 25, 2003

Waves, Optics, and Particles, Fall 2003

Homework Assignment # 11

(Due Thursday, December 4 at 5:00pm *sharp*.)

Agenda and readings for the week of December 1:

Skills to be mastered:

- Understand the wave nature of particles, be able to compute de Broglie wavelength for a particle of known energy, momentum or velocity
- Use the particle-wave correspondence to predict the results of simple scattering and "diffraction" experiments with electrons
- Understand and apply Heisenberg's uncertainty principle
- Understand the motion of a particle in a box from quantum mechanical point of view
- Be able to solve Schrödinger's equation for one-dimensional problems
- Use Schrödinger's equation and Feynman's "sum over histories" approach to study particle motion in one-dimensional potentials

Lectures and Readings:

Readings marked YF are from the text Young and Freedman, *University Physics*, 10th edition. Readings marked LN are from the course lecture notes to be found at http://people.ccmr.cornell.edu/~muchomas/P214.

- Lec 26, 12/02 (Tue): Schrödinger's equation and its solutions.
  Readings: YF 42-1, 42-2, 42-3, 42-4. LN "Quantum III: Particle in a box . . . ," Secs. 1.1–1.6; LN "Quantum IV: Scattering Theory," Sec. 4.
- Lec 27, 12/04 (Thu): Three Nobel Ideas
  Readings: LN "Feynman diagrams, . . . ," Secs. 3.4, 3.5, 4, 5.

Contents

1 Davisson-Germer Experiment

In this problem, we will consider the experiment that was first performed in 1927 by C. Davisson and L. Germer. This experiment has convincingly demonstrated the wave nature of electrons.

(a) An "electron gun" is a device in which electrons are created and accelerated by an electric field. The electrons are emitted by the source with very small (essentially zero) velocities. The potential difference between the source and the point where the electrons exit the gun is 750 V. Find the momentum of electrons as they exit the gun. An electron has a mass $m = 9.1 \cdot 10^{-31}$ kg and electric charge $e = -1.6 \cdot 10^{-19}$ C.

(b) According to de Broglie hypothesis, what is the wavelength of the electrons exiting the gun?

(c) A beam of electrons from the gun is directed at the surface of a nickel crystal. Electron velocities are perpendicular to the surface. The crystal is an array of equally spaced Ni atoms, with two neighboring atoms separated by $d = 0.1$ nm. Electrons hitting the atoms are reflected. Davisson and Germer studied how many electrons bounce off at different angles. Sketch the flux of electrons as a function of the angle θ. Find the values of θ at which the electron flux is maximal.

<u>*HINT:*</u> The crystal acts like a "reflection grating" which was studied in problem set # 9, problem 2.

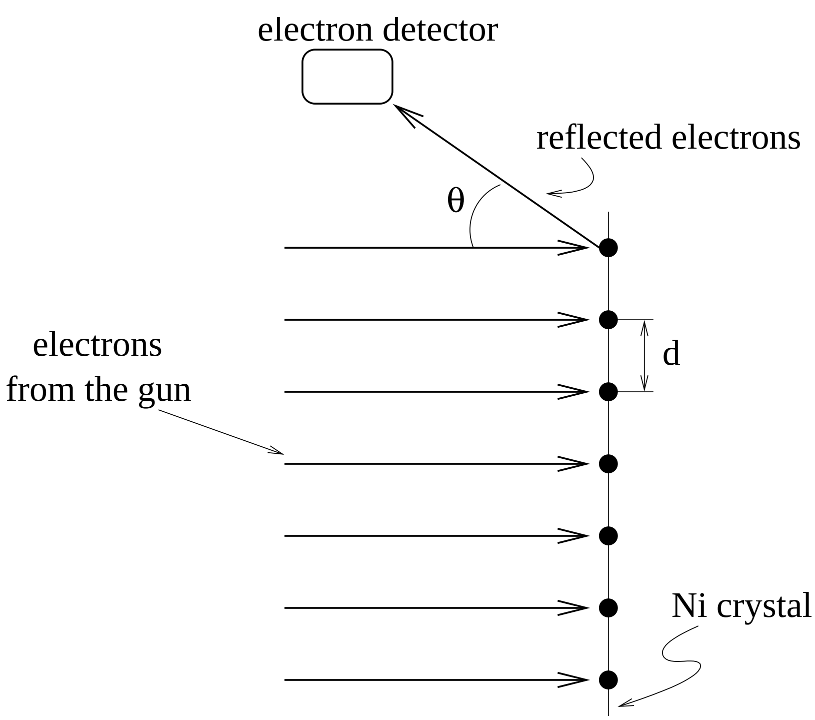

Figure 1: Davisson-Germer experiment.

2 Heisenberg Uncertainty Principle

A beam of electrons with momenta $p = 6.6 \cdot 10^{-24}$ N·sec is directed at a slit of width $a = 10$ nm. The electrons are then observed at a screen a distance $D = 10$ cm from the slit.

(a) What is the wavelength of the electrons? Is it smaller or larger than the width of the slit?

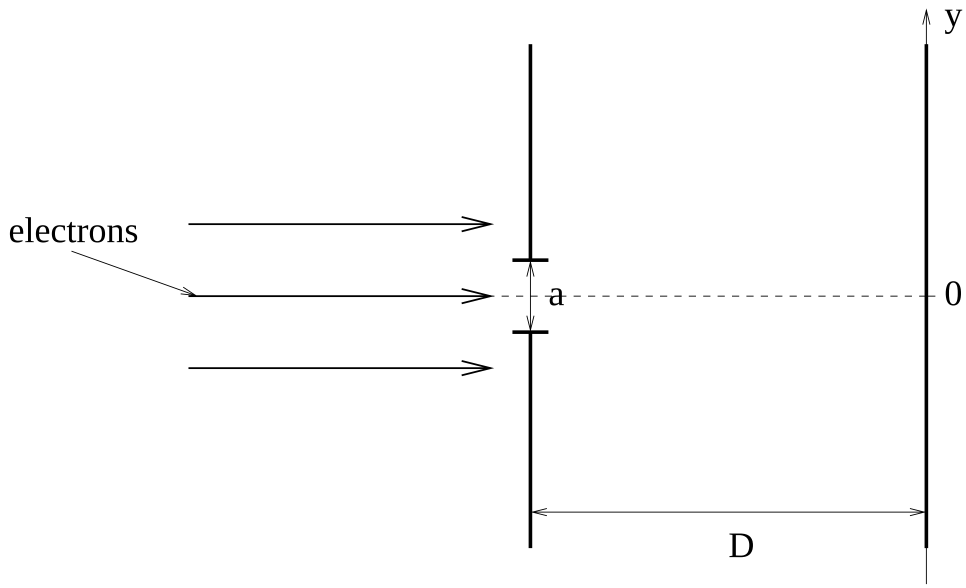

Figure 2: Electron diffraction experiment.

(b) Sketch the flux of electrons as a function of the position on the screen y. (The point opposite the center of the slit is at $y = 0$.) On the same sketch, show the flux you would expect if electrons did *not* behave as waves.

(c) How far from $y = 0$ can one place an electron detector and still detect a non-zero flux?

<u>*HINT:*</u> Recall problem set # 9, problem 5 ("Alone in the Dark".) Just like in that problem, you can assume that flux is zero outside the central intensity maximum!

(d) Electrons hitting the screen away from $y = 0$ have a non-zero momentum in the y direction. Find p_y of an electron hitting the screen at a point y. With the same assumption as in part (c), what is the *maximal* value of the magnitude of this momentum, $|p_y|$? What is its *minimal* value? The "uncertainty", or spread, in p_y is defined as $\Delta p_y = \max(|p_y|) - \min(|p_y|)$. Find Δp_y.

<u>*NOTE:*</u> Since the y component of the electron momentum does not change on the way from the slit to the screen, Δp_y can also be thought of as the uncertainty in the electron momentum at the moment when it passes through the screen.

(e) Observing an electron on the screen, we do not know exactly its y coordinate when it passed through the slit: it could be anywhere from $-a/2$ to $a/2$. Thus, the "uncertainty" in the position of the electron, Δy, is equal to the width of the slit a. Show that the uncertainties in the coordinate of the electron and its position satisfy the relation

$$\Delta y \Delta p_y = \hbar.$$

<u>*NOTE:*</u> Heisenberg uncertainty principle states that for *any* physical system, Δ(coordinate)Δ(momentum) is *at least* $\hbar/2$.

(f) If the slit is made very narrow, $a \to 0$, what is the expected uncertainty in momentum, Δp_y? Based on this result, sketch the expected flux of electrons as a function of the position on the screen for such a narrow slit. Does the result agree with the expectation from our study of wave interference pattern from one narrow slit?

3 From Billiards to Nuclear Physics

(a) Solve Young & Freedman, problem 42-1, parts (a), (b), and (c).

(b) Solve Young & Freedman, problem 42-2.

(c) Using the results in parts (a) and (b), explain why quantum mechanical effects are important for nuclear physics but not for the game of billiards.

4 Tunneling

A beam of particles of mass m and kinetic energy $E < eV_0$ enter from the right, traveling from right to left, toward a potential function $V(x)$ of shape shown in Fig. 3.

(a) What is the momentum p of the incoming particles?

(b) Taking $k = p/\hbar$, show that the wavefunction

$$\Psi_>(x) = e^{-ik(x-a)} + \underline{r}e^{ik(x-a)}$$

satisfies the Schrödinger's equation for $x > a$, while the wavefunction

$$\Psi_<(x) = \underline{t}e^{-ikx}$$

satisfies the Schrödinger's equation for $x < 0$. What is the physical meaning of $\underline{r}$ and $\underline{t}$?

Note: In lecture on Dec. 2, we will show that the Schrödinger equation (equation of motion for the wave function $\Psi(x)$) is

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} + V(x)\Psi(x) = E\Psi(x).$$

<u>*HINT:*</u> Recall problem set # 8, problem 1.

(c) Show that the wavefunction

$$\Psi_{\text{in}}(x) = Ce^{-\alpha x} + De^{\alpha x}$$

satisfies the Schrödinger's equation for $0 < x < a$ for any values of C, D. Determine α.

(d) Write down the equations that should be satisfied by the wavefunctions at $x = 0$ and $x = a$ in terms of the unknown coefficients C, D, $\underline{t}$, $\underline{r}$.
Note: In lecture on Dec. 2, we will show that the wave function $\Psi(x)$ *and* its derivative $d\Psi(x)/dx$ are always both continuous for finite potentials.

(e) By solving the equations obtained in part (d), find the coefficients $\underline{r}$ and $\underline{t}$. What is the probability that the incoming particle will pass the barrier (or "tunnel" through it)?
<u>*HINT:*</u> It is easiest to first solve for C and D in terms of $\underline{t}$ from the equation at $x = 0$, and then to substitute the result into the equation you get from the boundary conditions at $x = a$.

5 Potential Well

The setup is the same as in the previous problem, except the potential $V(x)$ now has the shape shown in the figure 4 ($V = +\infty$ for $x < 0$.) Again, the wavefunction at $x > a$ is given by

$$\Psi_>(x) = e^{-ik(x-a)} + \underline{r}e^{ik(x-a)}.$$

We will obtain the coefficient $\underline{r}$ using two different methods.

<u>*HINT:*</u> This problem is very similar to problem 2 on problem set # 8!

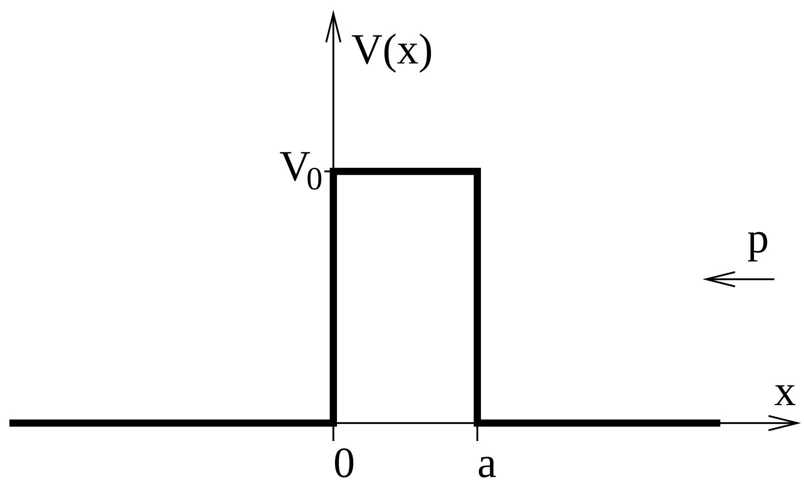

Figure 3: Tunneling.

5.1 Matching on the boundary

(a) Show that the function
$$\Psi_{\rm in}(x) = Ce^{-ik_0x} \; + \; De^{ik_0x}$$
satisfies the Schrödinger's equation for $0 < x < a$ for any values of C, D. Determine k_0.

(b) What is the boundary condition that should be satisfied by $\Psi_{\rm in}(x)$ at $x = 0$? Use this boundary condition to simplify $\Psi_{\rm in}(x)$.

(c) Write down the equations that should be satisfied by the wavefunctions at $x = a$. Solve these equations to obtain $\underline{r}$.

5.2 Sums over histories

As for waves on a string, we can define "transmission" and "reflection" coefficients at a boundary between regions (in our case, at $x = a$) with different potential energies and thus different wave vectors k_1 and k_2. For a wave going from Region 1 to Region 2 the coefficients are[1]

$$\begin{aligned} R_{1\to 2} &= \frac{k_1 - k_2}{k_1 + k_2} \\ T_{1\to 2} &= \frac{2k_1}{k_1 + k_2} \\ R_{2\to 1} &= \frac{k_2 - k_1}{k_1 + k_2} \\ T_{2\to 1} &= \frac{2k_2}{k_1 + k_2} \end{aligned}$$

(a) What is the wavefunction at $x = a$ of the wave that was reflected from the boundary of the potential well without entering into it? What about the wave that entered the potential well, was reflected from the infinite potential wall at $x = 0$, and then escaped from the well?

(b) Generalizing your result from part (a), obtain the wavefunction at $x = a$ of the wave moving from left to right in the region outside the potential well by summing over all possible back-and-forth reflections within the well. Use this result to obtain $\underline{r}$.

[1] These results will be derived in the lecture on 12/02. Compare them with the corresponding coefficients for a wave on a string!

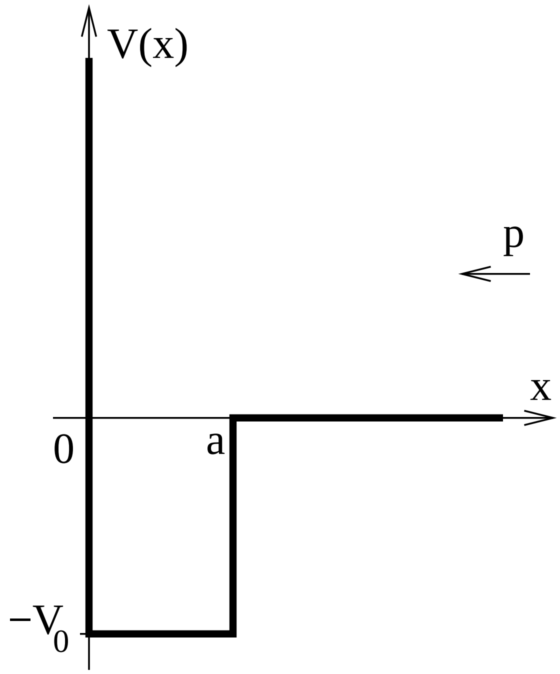

Figure 4: Potential well.

5.3 Getting Stuck

Using your results above, find the probability that an incoming particle will get stuck in the potential well.

HINT: First compute the probability that the particle will be reflected by the wall/well combination of this problem.