Physics 214 Fall 2003 PS#2

1) Identifying general solutions

$$-k(x-x_{eq}) = m\frac{d^2x}{dt^2}$$

a) $x(t) = x_{eq} + A\cos\omega_1 t$ $\quad \frac{dx}{dt} = -A\omega_1 \sin\omega_1 t$

$\frac{d^2x}{dt^2} = -A\omega_1^2 \cos\omega_1 t$ $\qquad -kA\cos\omega_1 t = -mA\omega_1^2\cos\omega_1 t$

$x(t)$ is a solution if $\omega_1^2 = k/m$, but it is not a general solution because there is only one free parameter, A.

b) $x(t) = C + A_1\sin\omega_1 t + A_2\cos\omega_1 t$ $\quad \frac{dx}{dt} = A_1\omega_1\cos\omega_1 t - A_2\omega_1\sin\omega_1 t$

$\frac{d^2x}{dt^2} = -A_1\omega_1^2\sin\omega_1 t - A_2\omega_1^2\cos\omega_1 t$

$-k(C + A_1\sin\omega_1 t + A_2\cos\omega_1 t - x_{eq}) = -m(A_1\omega_1^2\sin\omega_1 t + A_2\omega_1^2\cos\omega_1 t)$

$x(t)$ is a solution if $\omega_1^2 = k/m$ and $C = x_{eq}$. It is a general solution because there are two free parameters, A_1 and A_2.

c) $x(t) = x_{eq} + A_1\sin\frac{\omega_1}{2}t + A_2\cos\frac{\omega_1}{2}t$ $\quad \frac{dx}{dt} = A_1\frac{\omega_1}{2}\cos\frac{\omega_1}{2}t - A_2\frac{\omega_1}{2}\sin\frac{\omega_1}{2}t$

$\frac{d^2x}{dt^2} = -A_1\left(\frac{\omega_1}{2}\right)^2\sin\left(\frac{\omega_1}{2}t\right) - A_2\left(\frac{\omega_1}{2}\right)^2\cos\left(\frac{\omega_1}{2}t\right)$

$-k\left(A_1\sin\frac{\omega_1}{2}t + A_2\cos\frac{\omega_1}{2}t\right) = -m\left(A_1\left(\frac{\omega_1}{2}\right)^2\sin\left(\frac{\omega_1}{2}t\right) + A_2\left(\frac{\omega_1}{2}\right)^2\cos\left(\frac{\omega_1}{2}t\right)\right)$

$x(t)$ is a solution if $\left(\frac{\omega_1}{2}\right)^2 = k/m$. It is general because there are two free parameters, A_1 and A_2.

d) $x(t) = x_{eq} + A_1 e^{i\omega_0 t} - A_2 e^{-i\omega_0 t}$ $\quad \frac{dx}{dt} = A_1 i\omega_0 e^{i\omega_0 t} + A_2 i\omega_0 e^{-i\omega_0 t}$

$\frac{d^2x}{dt^2} = -A_1\omega_0^2 e^{i\omega_0 t} + A_2\omega_0^2 e^{-i\omega_0 t}$ $\qquad -k(A_1e^{i\omega_0 t} - A_2 e^{-i\omega_0 t}) = -m(A_1\omega_0^2 e^{i\omega_0 t} - A_2\omega_0^2 e^{-i\omega_0 t})$

$x(t)$ is a solution if $\omega_0^2 = k/m$. It is general because there are two free parameters, A_1 and A_2. However, it is not <u>physical</u> since x is complex. To get a physical, real solution, we would need $A_2 = -A_1$ – but then there is only <u>one</u> adj. parameter.

e) $x(t) = x_{eq} + A\cos[\omega_0(t-t_0)]$ $\quad \frac{dx}{dt} = -A\omega_0 \sin(\omega_0(t-t_0))$

$\frac{d^2x}{dt^2} = -A\omega_0^2\cos(\omega_0(t-t_0))$ $\quad -kA\cos(\omega_0(t-t_0)) = -mA\omega_0^2\cos(\omega_0(t-t_0))$

$x(t)$ is a solution if $\omega_0^2 = k/m$. It is general because there are two free parameters, A and t_0.

f) $x(t) = A_1 \sin\omega_1 t + A_2 \sin\omega_2 t$ $\quad \frac{dx}{dt} = A_1\omega_1\cos\omega_1 t + A_2\omega_2\cos\omega_2 t$

$\frac{d^2x}{dt^2} = -A_1\omega_1^2\sin\omega_1 t - A_2\omega_2^2\sin\omega_2 t$

$-k(A_1\sin\omega_1 t + A_2\sin\omega_2 t - x_{eq}) = -m(A_1\omega_1^2\sin\omega_1 t + A_2\omega_2^2\sin\omega_2 t)$

$x(t)$ is a solution if $x_{eq} = 0$ and $\omega_1^2 = \omega_2^2 = k/m$ $\Rightarrow$

$x(t) = (A_1 + A_2)\sin\sqrt{\frac{k}{m}}\,t$. It is not a general solution because there is only one free parameter, the sum of A_1 and A_2.

g) $x(t) = x_{eq} + B\tan\omega_0 t + C\sec\omega_0 t$

$\frac{dx}{dt} = B\omega_0\sec^2\omega_0 t + C\omega_0\sec\omega_0 t\tan\omega_0 t$

$\frac{d^2x}{dt^2} = 2B\omega_0^2\sec^2\omega_0 t\tan\omega_0 t + C\omega_0^2\sec^3(\omega_0 t) + C\omega_0^2\tan^2\omega_0 t\sec\omega_0 t$

$-k(B\tan\omega_0 t + C\sec\omega_0 t) = m(2B\omega_0^2\sec^2\omega_0 t\tan\omega_0 t + C\omega_0^2\sec\omega_0 t(\sec^2\omega_0 t + \tan^2\omega_0 t))$

Since B and C are constants, $x(t)$ is a solution only if $B = C = 0$. This is the trivial solution $x(t) = x_{eq}$, and it is not general because there are no adjustable parameters.

h) $x(t) = x_{eq} - A\cos(\phi_0 - \omega_0 t)$ $\quad \frac{dx}{dt} = -A\omega_0\sin(\phi_0 - \omega_0 t)$

$\frac{d^2x}{dt^2} = A\omega_0^2\cos(\phi_0 - \omega_0 t)$ $\quad -k(-A\cos(\phi_0 - \omega_0 t)) = mA\omega_0^2\cos(\phi_0 - \omega_0 t)$

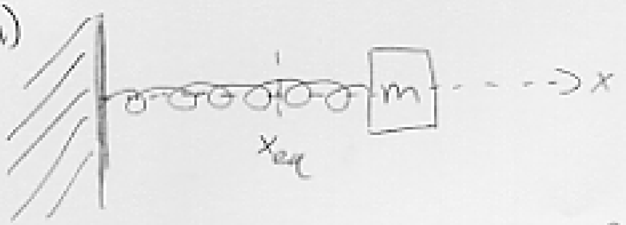
$x(t)$ is a solution if $\omega_0^2 = k/m$. It is general because there are two adjustable parameters, A and ϕ_0.

i) $\frac{d^2x}{dt^2} = -\frac{k}{m}(x - x_{eq})$

This isn't even a solution, it is a rewriting of the E.of M.

2) Damped Oscillator

a)



$$F_s = -k(x - x_{eq})$$

$$F_d = -bm\frac{dx}{dt}$$

Do these signs make sense? Suppose $x > x_{eq}$ and $\frac{dx}{dt} > 0$.
Then the mass is moving to the right, and away from x_{eq}, and both forces act to the left. OK! (Feel free to check the other cases.)

$$\Sigma F = ma = m\frac{d^2x}{dt^2} \qquad -k(x-x_{eq}) - bm\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

We can rearrange this as $\frac{d^2x}{dt^2} = -\omega_0^2(x-x_{eq}) - b\frac{dx}{dt}$, with $\omega_0 \equiv \sqrt{k/m}$.

b) $x(t) = x_{eq} + Ae^{-bt/2}\sin(\omega_1 t + \delta_0)$

$$\frac{dx}{dt} = A\omega_1 e^{-bt/2}\cos(\omega_1 t + \delta_0) - A\frac{b}{2}e^{-bt/2}\sin(\omega_1 t + \delta_0)$$
$$= -\frac{b}{2}(x - x_{eq}) + \omega_1 Ae^{-bt/2}\cos(\omega_1 t + \delta_0)$$

$$\frac{d^2x}{dt^2} = -\frac{b}{2}\frac{dx}{dt} - \omega_1^2 Ae^{-bt/2}\sin(\omega_1 t + \delta_0) - \omega_1\frac{b}{2}Ae^{-bt/2}\cos(\omega_1 t + \delta_0)$$
$$= -\frac{b}{2}\left(-\frac{b}{2}(x - x_{eq}) + \omega_1 Ae^{-bt/2}\cos(\omega_1 t + \delta_0)\right)$$
$$- \omega_1^2(x - x_{eq}) - \omega_1 b/2\, Ae^{-bt/2}\cos(\omega_1 t + \delta_0)$$
$$= \left(\frac{b^2}{4} - \omega_1^2\right)(x - x_{eq}) - b\omega_1 Ae^{-bt/2}\cos(\omega_1 t + \delta_0)$$

$$\cancel{-k(Ae^{-bt/2}\sin(\omega_1 t + \delta_0)) - bm\left(-\frac{b}{2}(x - x_{eq}) + \omega_1 Ae^{-bt/2}\cos(\omega_1 t + \delta_0)\right)}$$
$$\cancel{= m\left(\frac{b^2}{4} - \omega_1^2\right)(x - x_{eq})}$$

To check:

$$\left(\frac{b^2}{4} - \omega_1^2\right)(x - x_{eq}) - b\omega_1 \cancel{Ae^{-bt/2}\cos(\omega_1 t + \delta_0)} = -\omega_0^2(x - x_{eq})$$
$$+ \frac{b^2}{2}(x - x_{eq}) - b\omega_1 \cancel{Ae^{-bt/2}\cos(\omega_1 t + \delta_0)}$$

$x(t)$ is a solution if $\frac{b^2}{4} - \omega_1^2 = -\omega_0^2 + \frac{b^2}{2} \Rightarrow \omega_1 = \sqrt{\omega_0^2 - \frac{b^2}{4}}$

It is a general solution because there are two adjustable parameters, A and δ_0.

3) A bouncy ride

$m = 1000\,kg$ compression $= 2.5\,cm = 2.5\times10^{-2}\,m$. $v_0 = 10\,\frac{cm}{s} = 0.10\,m/s$

a) Before $t=0$, the system is in equilibrium. The force due to the compression of the springs is equal in magnitude and opposite in direction to the gravitational force:

$-k_{eff}(-2.5\times10^{-2}m) - mg = 0 \Rightarrow k_{eff} = \frac{1000\,kg(9.81\,m/s^2)}{2.5\times10^{-2}m} = 3.9\times10^5\,\frac{N}{m}$

b) 

$F_g = -mg$ $\qquad \Sigma F = m\frac{d^2x}{dt^2}$

$F_s = -k(x - x_{eq})$ $\qquad -mg - k(x - x_{eq}) = m\frac{d^2x}{dt^2}$

At equilibrium, $-mg - k(x - x_{eq}) = 0$.

If we define the zero of our x-axis to be at the equilibrium position, $-mg - k(-x_{eq}) = 0 \Rightarrow mg = kx_{eq}$.

Therefore, the equation of motion simplifies to $-kx = m\frac{d^2x}{dt^2}$.

c) From the lecture notes, we know that $x(t) = \Re(\underline{A}e^{i\omega_0 t})$ is a general solution to this E of M, with $\omega_0 \equiv \sqrt{k/m}$. Use the initial conditions to find the particular solution for $t \geq 0$.

$0 = x(0) = \Re(\underline{A}) = A_r$ $\qquad \frac{dx}{dt} = \Re(\underline{A}i\omega_0 e^{i\omega_0 t})$

$0.1 = \frac{dx}{dt}\Big|_{t=0} = \Re(\underline{A}i\omega_0) = \Re((A_r + iA_i)i\omega_0) = -A_i\omega_0$

$x(t) = \Re\left(\left(0 + i\left(-\frac{0.1}{\omega_0}\right)\right)e^{i\omega_0 t}\right) = \Re\left(\frac{0.1}{\omega_0}e^{i(\omega_0 t - \pi/2)}\right)$

$= \frac{0.1}{\omega_0}\cos(\omega_0 t - \pi/2)$ $\qquad \omega_0 = \sqrt{\frac{3.9\times10^5}{1000}} \simeq 20\,\frac{1}{sec}$

$\approx (5\times10^{-3}\,m)\sin(20t)$

d) From above, the amplitude is $\sim \frac{1}{2}$ a cm.

4a) $-k(x-x_{eq}) = m\frac{d^2x}{dt^2}$ $\qquad -\omega_0^2(x-x_{eq}) = \frac{d^2x}{dt^2}$

$$x(t) = x_{eq} + \frac{1}{2}\left[\underline{A}e^{i\omega_0 t} + \underline{A}^* e^{-i\omega_0 t}\right]$$

$$\frac{dx}{dt} = \frac{1}{2}i\omega_0\left[\underline{A}e^{i\omega_0 t} - \underline{A}^* e^{-i\omega_0 t}\right] \qquad \frac{d^2x}{dt^2} = -\frac{1}{2}\omega_0^2\left[\underline{A}e^{i\omega_0 t} + \underline{A}^* e^{-i\omega_0 t}\right]$$
$$= -\omega_0^2(x(t)-x_{eq}).$$

$x(t) = x_{eq} + \frac{1}{2}\left[\underline{A}e^{i\omega_0 t} + \underline{A}^* e^{-i\omega_0 t}\right]$ is a solution. It is a general solution because it has two real free parameters, A_r and A_i, where $\underline{A} \equiv A_r + iA_i$ and $\underline{A}^* \equiv A_r - iA_i$.

4b) $x(t) = x_{eq} + \frac{1}{2}\left[\underline{A}e^{i\omega_0 t} + \underline{A}^* e^{-i\omega_0 t}\right]$

$$x_0 = x(0) = x_{eq} + \frac{1}{2}\left[\underline{A} + \underline{A}^*\right]$$

$$\frac{dx}{dt} = \frac{1}{2}i\omega_0\left[\underline{A}e^{i\omega_0 t} - \underline{A}^* e^{-i\omega_0 t}\right] \qquad v_0 = \left.\frac{dx}{dt}\right|_{t=0} = \frac{1}{2}i\omega_0\left[\underline{A} - \underline{A}^*\right]$$

$$x_0 - x_{eq} = \frac{1}{2}(\underline{A} + \underline{A}^*)$$
$$\frac{v_0}{i\omega_0} = \frac{1}{2}(\underline{A} - \underline{A}^*) \quad \Rightarrow \quad \underline{A} = x_0 - x_{eq} + \frac{v_0}{i\omega_0} = x_0 - x_{eq} - \frac{iv_0}{\omega_0}$$
$$\underline{A}^* = x_0 - x_{eq} - \frac{v_0}{i\omega_0} = x_0 - x_{eq} + \frac{iv_0}{\omega_0}$$

5a) $-k(x-x_{eq}) - bm\frac{dx}{dt} = m\frac{d^2x}{dt^2}$

Try $x(t) = x_{eq} + \mathcal{R}(\underline{A}e^{\alpha t})$ $\qquad \frac{dx}{dt} = \mathcal{R}(\underline{A}\alpha e^{\alpha t})$ $\qquad \frac{d^2x}{dt^2} = \mathcal{R}(\underline{A}\alpha^2 e^{\alpha t})$

$$\Rightarrow -k\mathcal{R}(\underline{A}e^{\alpha t}) - bm\mathcal{R}(\underline{A}\alpha e^{\alpha t}) = m\mathcal{R}(\underline{A}\alpha^2 e^{\alpha t})$$

$x(t) = x_{eq} + \mathcal{R}(\underline{A}e^{\alpha t})$ is a solution if $-k - bm\underline{\alpha} = m\underline{\alpha}^2$

$$\Rightarrow \underline{\alpha} = \frac{-bm \pm \sqrt{b^2m^2 - 4mk}}{2m} = -\frac{b}{2} \pm \frac{1}{2}i\sqrt{4\omega_0^2 - b^2}.$$

(Here we have assumed that $b < 2\sqrt{k/m}$, and let $\omega_0 = \sqrt{k/m}$.)

$$x(t) = x_{eq} + \mathcal{R}\left(\underline{A}e^{-\frac{b}{2}t}e^{\pm\frac{1}{2}\sqrt{4\omega_0^2-b^2}\,t}\right)$$
$$= x_{eq} + \mathcal{R}\left((A_r + iA_i)e^{-\frac{b}{2}t}\left(\cos\left(\pm\tfrac{1}{2}\sqrt{4\omega_0^2-b^2}\right) \pm i\sin\left(\pm\tfrac{1}{2}\sqrt{4\omega_0^2-b^2}\right)\right)\right)$$
$$= x_{eq} + A_r e^{-\frac{b}{2}t}\cos\left(\pm\tfrac{1}{2}\sqrt{4\omega_0^2-b^2}\right) \mp A_i e^{-\frac{b}{2}t}\sin\left(\pm\tfrac{1}{2}\sqrt{4\omega_0^2-b^2}\right)$$

We can absorb the $\mp$ into our definition of A_i, so we have $x(t) = x_{eq} + A_r e^{-\frac{b}{2}t}\cos\left(\frac{t}{2}\sqrt{4\omega_0^2-b^2}\right) + A_i e^{-\frac{b}{2}t}\sin\left(\frac{t}{2}\sqrt{4\omega_0^2-b^2}\right)$, which is a real general solution with two free parameters, A_r and A_i.

We can make this look familiar by defining new parameters $A \equiv \sqrt{A_r^2+A_i^2}$ and ϕ such that

$A\cos\phi = A_r$ and $A\sin\phi = A_i$.

Then $x(t) = x_{eq} + Ae^{-bt/2}\left(\cos\frac{t}{2}\sqrt{4\omega_0^2-b^2}\,\cos\phi + \sin\frac{t}{2}\sqrt{4\omega_0^2-b^2}\,\sin\phi\right)$

$$= x_{eq} + Ae^{-bt/2}\cos\left(\frac{t}{2}\sqrt{4\omega_0^2-b^2} - \phi\right).$$

This is the form of the general solution on p. 412 of the text, if one defines $\phi_{text} = -\phi_{\text{defined above}}$. Similarly, if we define $\delta_0 = -\phi + \pi/2$, we have the version of the general solution from problem 2b):

$$x(t) = x_{eq} + Ae^{-bt/2}\sin(\omega_1 t + \delta_0), \text{ where}$$

$$\omega_1 = \frac{1}{2}\sqrt{4\omega_0^2-b^2} = \sqrt{\omega_0^2 - b^2/4}, \text{ as before.}$$

5b) $k = 4\times10^5\ \mathrm{N/m}$, $m = 1000\ \mathrm{kg}$, $b = 2\ \mathrm{s}^{-1}$

$x_{eq} = 0$, $x_0 = 0$, $v_0 = 0.1\ \mathrm{m/s}$

$0 = x(0) = 0 + A\sin\delta_0 \Rightarrow \delta_0 = 0$

$\frac{dx}{dt} = A\omega_1 e^{-bt/2}\cos(\omega_1 t + \delta_0) - \frac{Ab}{2}e^{-bt/2}\sin(\omega_1 t + \delta_0)$

$0.1 = \left.\frac{dx}{dt}\right|_{t=0} = A\omega_1 \qquad \omega_1 = \sqrt{\frac{4\times10^5}{1000} - 1} \approx 20\ \frac{1}{\mathrm{sec}}$

$A \approx 5\times10^{-3}\ \mathrm{m}$.

$x(t) = (.5\ \mathrm{cm})\,e^{-t}\sin\left(20\tfrac{\mathrm{rad}}{\mathrm{s}}\,t\right)$