Cornell University

Department of Physics

Phys 214 September 8, 2003

Waves, Optics, and Particles, Fall 2003

Homework Assignment # 3

(Due Thursday, September 18 at 5:00pm *sharp*.)

Agenda and readings for the week of September 15:

Skills to be mastered:

- be able to use complex representation to solve differential equations;
- be able to convert between wavelength (λ), wavenumber or spatial frequency ($\kappa = 1/\lambda$), and propagation number ($k = 2\pi/\lambda$);
- given the wave speed (v) and any one of the quantities (T, f, ω), be able to find any one of (λ, κ, k), and vice versa;
- given any one of the quantities (T, f, ω) AND any one of the quantities (λ, κ, k), be able to find v;

Lectures and Readings:

Readings marked YF are from the text Young and Freedman, *University Physics*, 10th edition. Readings marked LN are from the course lecture notes to be found at http://people.ccmr.cornell.edu/~muchomas/P214.

- Lec 4, 09/16 (Tue): Normal modes; boundary conditions for string; dispersion relation $v = \sqrt{(\tau/\mu)}$.
Readings: LN "Intro to Waves: Waves on a String and Standing Waves," Sec. 4.3–5; YF 19-4, 19-5.
- Lec 5, 09/18 (Thu): Wave equation for sound.
Readings: LN " Other Waves: Sound," Sec. 1–2; YF 19-6, 19-7.

Contents

1 Closer look at the response of driven oscillators

In lecture we derived an expression for the complex amplitude $\underline{A} = Ae^{i\phi_o}$ of a driven and damped harmonic oscillator, with *real amplitude* A and *initial phase* ϕ_o,

$$A = |\underline{A}| = \frac{\frac{F_o}{m}}{\sqrt{(\omega_o^2 - \omega^2)^2 + (b\omega)^2}} \qquad \text{and} \qquad \phi_o = \tan^{-1}\left(\frac{b\omega}{\omega^2 - \omega_o^2}\right) \quad , \tag{1}$$

where $\omega_o \equiv \sqrt{k/m}$ is the natural frequency of the system.

In order to better appreciate the physical significance of these quantities and their frequency dependence, it is useful to *plot* them versus frequency and to identify certain important parameters of these plots. In this problem you will do this, starting from analyzing $A(\omega)/\frac{F_o}{m}$ and $\phi_o(\omega)$ in various limits[1]:

(a) Evaluate $A(\omega = 0)/\frac{F_o}{m}$ and $\phi_o(\omega = 0)$.

(b) Evaluate $\lim_{\omega\to\infty} A(\omega)/\frac{F_o}{m}$ and $\lim_{\omega\to\infty} \phi_o(\omega)$.

(c) Show that $A(\omega)/\frac{F_o}{m}$ has a *maximum* at $\omega = \omega_R = \sqrt{\omega_o^2 - b^2/2}$ if $b^2 < 2\omega_o^2$. Express the value of $(A/\frac{F_o}{m})$ at the maximum in terms of ω_o and b.

Hint: Argue that the ratio of two positive expressions has a maximum when the denominator has a minimum. Then show that the denominator of A in eq. (1) has a minimum at $\omega = \omega_R$ by checking that $\frac{dD(\omega)}{d\omega}\Big|_{\omega=\omega_R} = 0$ for the function $D(\omega) \equiv (\omega_o^2 - \omega^2)^2 + (b\omega)^2$.

(d) In the limit $b \ll \omega_o$, show that $\omega_R \approx \omega_o$ and that $A(\omega)/\frac{F_o}{m}$ is *approximately* equal to $(A/\frac{F_o}{m})_{\text{max}}/\sqrt{2}$ at each of the frequencies $\omega_- = \omega_o - b/2$ and $\omega_+ = \omega_o + b/2$. Compute the 'frequency band-width' $\Delta\omega \equiv \omega_+ - \omega_-$.

(e) <u>Sketch</u> plots of the functions $A(\omega)/\frac{F_o}{m}$ and $\phi_o(\omega)$ in the limit $b \ll \omega_o$. Label the quantities ω_o, $(A/\frac{F_o}{m})_{\text{max}}$, and $\Delta\omega$ on your plot.
Note: For this and other plots, you may plot the function on a plotting calculator or computer and then just sketch the basic shape of what you see. (You need not obtain a printout.) However, it is important that you label key points on your sketches (such as the position, width and height of the maxima) in terms of the basic quantities in the problem.

2 Application: care crossing bridges

To avoid the possibility of driving bridges into large oscillations by the periodic impact of footsteps, the US Army infantry now never marches instep across bridges. In this problem, we will evaluate whether this concern is realistic by computing how closely timed the footsteps must be to match a bridge's natural frequency. To answer this question, we shall use a simple model of the bridge as a driven, damped oscillator.

Within this model, we take the march steps to correspond to a periodic driving force $F = F_o\cos(\omega t)$, where $\omega = \frac{2\pi}{T_{\text{st}}}$ with T_{st} being the time between the footsteps of a unit marching in formation. To make the model realistic, we assume the following information:

- We take the bridge to have a mass m and spring constant k such that the bridge has a natural angular frequency $\omega_0 = 2\pi/T_o$, where $T_o \equiv 0.5$ sec is something close to the period of a march step (time between single steps).
- We assume that the natural damping of the bridge's motion is such that, if initially put in motion but then left undriven, the bridge will noticeably vibrate up and down about 20 times. Specifically, take the damping parameter b to be such that the amplitude of *undriven* oscillations decays to $1/e$ of its original value in a time $t = 10T_o$

[1]We will be interested in the *normalized* plot $A(\omega)/\frac{F_o}{m}$ rather that in $A(\omega)$, because the former clearly shows how the amplitude of the driven oscillations compares with the amplitude of the driving force.

(a) Use the above information to <u>extract</u> the numerical parameters (ω_o, b) needed to make a computer or calculator plot of $A/(\frac{F_o}{m})$ versus the marching frequency ω, where A is the amplitude of the bridge's oscillation. <u>Turn in</u> a properly labeled sketch or a copy of the resulting plot.

(b) Read off from your plot the range within which $T_{\rm st}$ must fall in order for the amplitude to be within 1/2 of its maximum value. (An *approximate* range read off from your plot is fine.)

(c) <u>Comment</u> in a brief sentence or two on whether you feel the concern about driving such a bridge into resonance is realistic.

3 Energy in oscillators

(a) From the general solution

$$x(t) = x_{\rm eq} + \Re\left(\underline{A}e^{i\omega_o t}\right)$$

<u>determine</u> the total (potential plus kinetic) energy of an *undamped* oscillator in terms of $|\underline{A}|^2$, m and ω_o.
Hint: Use the fact that $k = m\omega_o^2$.

(b) From the general solution

$$x(t) = x_{\rm eq} + A\cos\left(\omega_o t + \phi_o\right)$$

<u>compute</u> the *average* kinetic energy and the *average* potential energy of the undamped oscillator in terms of A^2, *and* <u>show</u> that they are equal.
Hint: The average values of both $\sin^2$ and $\cos^2$ are 1/2.

(c) Using the fact that the work done per unit time (power) against any force is $P = \vec{F}\cdot\vec{v}$, <u>show</u> that for the drag force described in lecture, the total energy E (kinetic plus potential) of a damped oscillator obeys

$$\frac{dE}{dt} = -bmv^2. \tag{2}$$

How does the average of this energy loss dE/dt compare to the average kinetic energy?

(d) If the damping b is relatively small, the result in **(b)** that the average kinetic energy is the same as the average potential energy is a very good approximation. Under this approximation, use your result in **(c)** to *show* that *on average*

$$\frac{dE}{dt} \approx -bE. \tag{3}$$

(e) The solution to the equation in **(d)** is that the energy decays exponentially as

$$E(t) = E_o e^{-bt}.$$

<u>Explain</u> (in a brief sentence or two) why the exponent here is "$-bt$" whereas the exponent in the general solution for damped harmonic motion is "$-bt/2$".

4 Other types of damping

An object of mass m is attached to an ideal spring (k) and driven by an external force $F_{\rm ext}(t) = F_0\cos(\omega t)$. The drag force exerted on the object is given by $F_{\rm drag} = -cm\frac{d^3x}{dt^3}$.

(a) <u>Find</u> the (real) amplitude A of oscillations.

Hint: As in class, assume that the oscillations are harmonic and use the complex representation.

(b) For $\omega_o = 100$ sec^{-1} and $c = \frac{1}{2\omega_o}$, <u>sketch</u> $A/\frac{F_o}{m}$ versus ω.

5 Conversion of wave quantities

(a) Many cordless phones transmit and receive 900 MHz microwaves. Find the wavelength λ, the wavenumber κ, and propagation number k. Microwaves travel through air at speed $c = 3.0 \times 10^8$ m/s.

(b) Microwave ovens generally operate at a frequency of $f = 2.5$ GHz. Compute the wavelength of the waves inside an oven. In a brief sentence or two, use your answer to explain the hot and cold spots on a reheated pizza in a cheap microwave oven without a rotating dish.

(c) Do problem YF 19-2 (p. 615).
Note: The normal range of human hearing is 20 Hz – 20,000 Hz.