Phys 214 HW#3

① a) $\omega \ll \omega_0$

$$A(\omega)/\tfrac{F_0}{m} = \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+(b\omega)^2}}$$

$$A(\omega=0)/\tfrac{F_0}{m} = \frac{1}{\sqrt{(\omega_0^2-0^2)^2+(0b)^2}}$$

$$= \frac{1}{\sqrt{\omega_0^4}} = \frac{1}{\omega_0^2}$$

Amplitude is positive.

$$\phi_0(\omega) = \tan^{-1}\left(\frac{b\omega}{\omega^2-\omega_0^2}\right)$$

$$\phi_0(\omega=0) = \tan^{-1}\left(\lim_{\omega\to 0}\left(\frac{b\omega}{\omega^2-\omega_0^2}\right)\right)$$

$$= \tan^{-1}\left(\lim_{\omega\to 0}\left(\frac{b\omega}{-\omega_0^2}\right)\right) \text{ by L'Hopital's rule}$$

$$= \tan^{-1} 0 = 0 \text{ or } \pi$$

$\omega > 0$, $\omega^2-\omega_0^2 > 0$

$\Rightarrow$ choose 0

b) $\omega_0 \ll \omega$

$$\lim_{\omega\to\infty} A(\omega)/f = \lim_{\omega\to\infty}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+(b\omega)^2}}$$

b/c $\omega_0 \ll \omega$ when $\omega \to \infty$ $\approx \dfrac{1}{\sqrt{(-\omega^2)^2+b^2\omega^2}}$

$$= \frac{1}{\sqrt{\omega^4+b^2\omega^2}}$$

$$= \frac{1}{\omega^2} \quad \text{b/c } \omega\to\infty \Rightarrow \omega^2 \ll \omega^4$$

$$\lim_{\omega\to\infty}\phi_0(\omega) = \lim_{\omega\to\infty}\tan^{-1}\left(\frac{b\omega}{\omega^2-\omega_0^2}\right)$$

$$= \tan^{-1}\left(\lim_{\omega\to\infty}\frac{b\omega}{\omega^2-\omega_0^2}\right)$$

$$= \tan^{-1}\left(\lim_{\omega\to\infty}\frac{b}{2\omega}\right) \text{ by L'Hopital's rule}$$

$$= \tan^{-1} 0 = 0 \text{ or } \pi$$

$\Rightarrow$ choose π

c) $A(\omega)/\tfrac{F_0}{m} = \left[(\omega_0^2-\omega^2)^2+(b\omega)^2\right]^{-1/2}$ written in a form for easy derivative taking

$$\frac{\partial}{\partial\omega}\left[A(\omega)/\tfrac{F_0}{m}\right] = -\tfrac{1}{\cancel{2}}\left[(\omega_0^2-\omega^2)^2+(b\omega)^2\right]^{-3/2}\left[\cancel{2}(\omega_0^2-\omega^2)(-2\omega)+\cancel{2}b^2\omega\right]$$ applying the chain rule

$$0 = \left[(\omega_0^2-\omega^2)^2+b^2\omega^2\right]^{-3/2}\left[-2\omega(\omega_0^2-\omega^2)+b^2\omega\right]$$ simplifying and setting equal to zero.

$$\Rightarrow \left[(\omega_0^2-\omega^2)^2+b^2\omega^2\right]^{-3/2}=0 \quad \text{or} \quad \omega\left[b^2-2(\omega_0^2-\omega^2)\right]=0$$

$\Downarrow$ $\omega = \infty$ which we can tell is a minimum by part b

$\omega = 0$ which turns out to be a minimum if $b^2 < 2\omega_0^2$

or $\omega_0^2-\omega^2 = \frac{b^2}{2}$

$\Rightarrow \omega = \sqrt{\omega_0^2 - b^2/2}$ which is real if $b^2 < 2\omega_0^2$

c) cont.

We must now show the $\omega = \omega_R \equiv \sqrt{\omega_0^2 - b^2/2}$ is a maximum

To do this we must show that $\left.\frac{\partial^2}{\partial \omega^2}\left[A(\omega)/\tfrac{F_0}{m}\right]\right|_{\omega=\omega_R} < 0$

As calculated above

$$\frac{\partial}{\partial \omega}\left[A(\omega)/\tfrac{F_0}{m}\right] = \frac{-\omega\left[b^2 - 2(\omega_0^2-\omega^2)\right]}{\left((\omega_0^2-\omega^2)^2 + b^2\omega^2\right)^{3/2}}$$

$$\frac{\partial^2}{\partial \omega^2}\left[A(\omega)/\tfrac{F_0}{m}\right] = \frac{3\omega\left[b^2 - 2(\omega_0^2-\omega^2)\right]^2}{\left[(\omega_0^2-\omega^2)^2 + b^2\omega^2\right]^{5/2}} - \frac{4\omega^2 + b^2(\omega_0^2-\omega^2)}{\left[(\omega_0^2-\omega^2)^2 + b^2\omega^2\right]^{3/2}}$$

$$= \frac{3\omega\left[b^2 - 2(\omega_0^2-\omega^2)\right]^2 - \left[4\omega^2 + b^2(\omega_0^2-\omega^2)\right]\left[(\omega_0^2-\omega^2)^2 + b^2\omega^2\right]}{\left[(\omega_0^2-\omega^2)^2 + b^2\omega^2\right]^{5/2}}$$

$\left[(\omega_0^2-\omega^2)^2 + b^2\omega^2\right]^{5/2} > 0$ because $(\omega_0^2-\omega^2)^2 > 0$ and $b^2\omega^2 > 0$

So we must show that the numerator is < 0

$$\text{numerator}\Big|_{\omega=\omega_R} = 3\omega_R\underbrace{\left[b - 2(b^2/2)\right]^2}_{0} - \left[\underbrace{4\omega_R^2}_{>0} + \underbrace{b^2\left(\tfrac{b^2}{2}\right)}_{>0}\right]\left[\underbrace{\tfrac{b^4}{4}}_{>0} + \underbrace{b^2\omega^2}_{>0}\right]$$

$= 0$ − positive #

$=$ a negative #

$\Rightarrow \omega = \omega_R$ is a maximum

$$\boxed{A(\omega=\omega_R)/\tfrac{F_0}{m} = \frac{1}{\sqrt{(\omega_0^2-\omega_R^2)^2 + b^2\omega_R^2}} = \frac{1}{\sqrt{b^4/4 + b^2(\omega_0^2 - \frac{b^2}{2})}} = \frac{1}{b\sqrt{\omega_0^2 - b^2/4}}}$$

d) $\omega_R = \sqrt{\omega_0^2 - b^2/2} = \omega_0\sqrt{1 - \frac{1}{2}\left(\frac{b}{\omega_0}\right)^2}$

$b \ll \omega_0 \Rightarrow \left(\frac{b}{\omega_0}\right)^2 \approx 0 \Rightarrow \omega_R \approx \omega_0$

$A/\frac{F_0}{m} = \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2 + b^2\omega^2}}$ So $A(\omega = \omega_0 \pm \frac{b}{2})/\frac{F_0}{m} = \frac{1}{\sqrt{(\mp b\omega_0 - \frac{b^2}{4})^2 + b^2(\omega_0^2 \pm b\omega_0 + \frac{b^4}{4})}}$

$$= \frac{1}{b\sqrt{\omega_0^2 \pm \frac{b\omega_0}{2} + \frac{b^4}{16} + \omega_0^2 \pm b\omega_0 + \frac{b^4}{4}}};$$

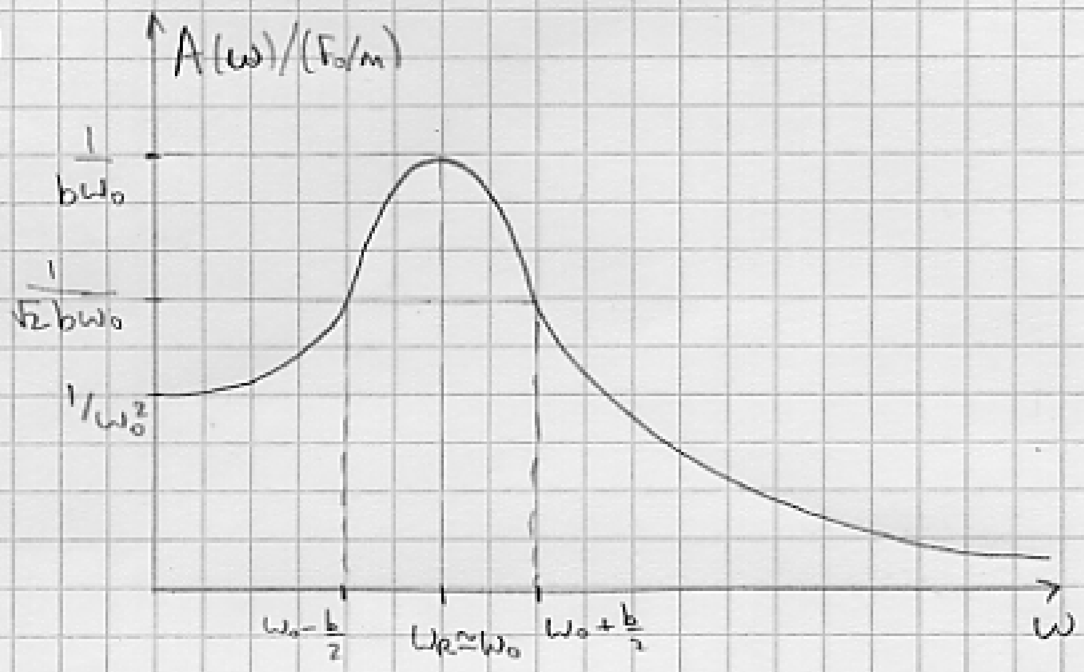
$$\approx \frac{1}{b\sqrt{2\omega_0^2}} = \frac{(A/\frac{F_0}{m})_{max}}{\sqrt{2}}$$

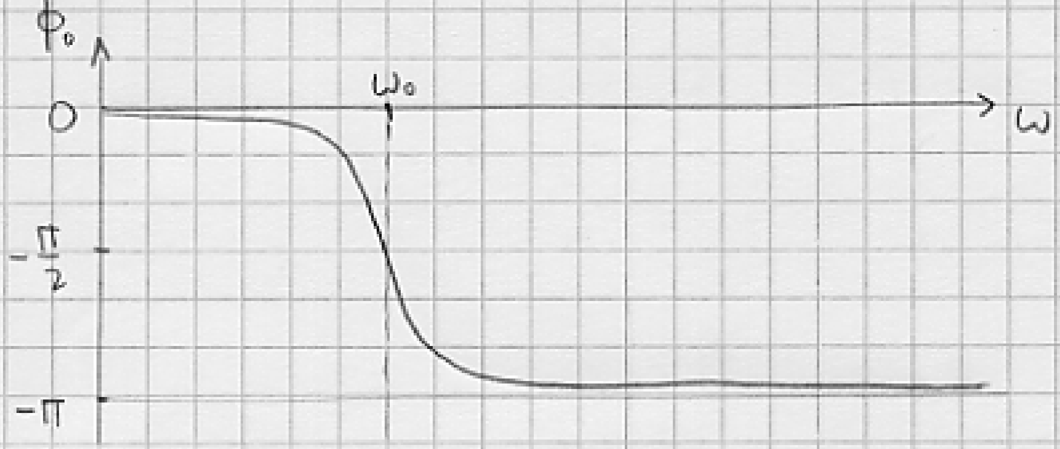
we ignore all higher order terms

$\Delta\omega = \omega_+ - \omega_-$

$= \omega_0 + \frac{b}{2} - (\omega_0 - \frac{b}{2}) = b$

e)

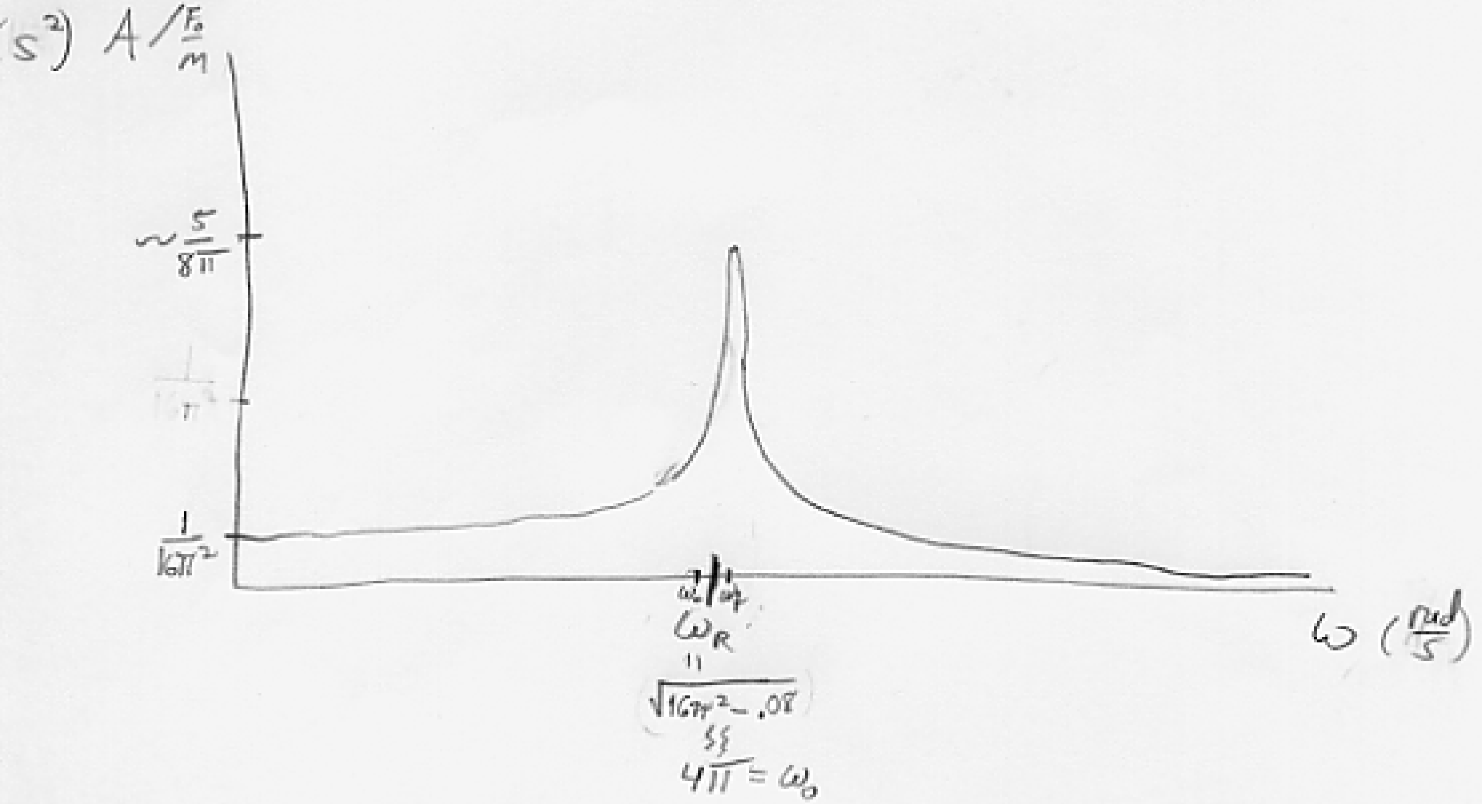

$A(\omega)/(F_0/m)$
$\frac{1}{b\omega_0}$
$\frac{1}{\sqrt{2}b\omega_0}$
$1/\omega_0^2$
$\omega_0-\frac{b}{2}$
$\omega_R\simeq\omega_0$
$\omega_0+\frac{b}{2}$
ω


ϕ_0
0
ω_0
ω
$-\frac{\pi}{2}$
$-\pi$

(2) a) $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{.5s} = 4\pi \frac{rad}{s}$

From HW#2, we know that the amplitude of a damped oscillator goes like $\sim Ae^{-bt/2}$

So $\frac{1}{e} = \frac{e^{-b(t+10T_0)/2}}{e^{-bt/2}} = e^{-5bT_0} \Rightarrow 5bT_0 = 1 \Rightarrow b = \frac{1}{5T_0} = \frac{2}{5} s^{-1}$



b) $\omega_0 - b \lesssim \omega \lesssim \omega_0 + b$

or

$4\pi - .4 \lesssim \omega \lesssim 4\pi + .4$

$12.16 \frac{rad}{s} \lesssim \omega \lesssim 12.96 \frac{rad}{s}$

c) This range of frequencies, although small is still significant. It is not unlikely that the soldiers will be marching at one of these frequencies. Therefore it may be worthwhile for the soldiers to break step.

③ a) $x(t) = x_{eq} + \mathrm{Re}(\underline{A}\, e^{i\omega_0 t})$, $\dot{x}(t) = \mathrm{Re}(\underline{A} i\omega e^{i\omega_0 t})$

pot. E $\quad V(t) = \frac{1}{2}K(x(t))^2 = \frac{1}{2}m\omega_0^2\left[\mathrm{Re}(\underline{A}e^{i\omega_0 t})\right]^2$

$= \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega_0 t+\phi)$ where $\underline{A} = Ae^{i\phi}$

Kin. E. $\quad K(t) = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\left[\mathrm{Re}(i\omega\underline{A}e^{i\omega t})\right]^2$

$= \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t+\phi)$

$E = \frac{1}{2}m\omega_0^2 A^2(\sin^2(\omega_0 t+\phi) + \cos^2(\omega_0 t+\phi)) = \frac{1}{2}m\omega_0^2 A^2$

b) $V_{avg} = \frac{1}{2\pi}\int_0^{2\pi} V(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}m\omega_0^2 A^2\cos^2(\omega_0 t+\phi) = \frac{1}{4}m\omega_0^2 A^2$

$K_{avg} = \frac{1}{2\pi}\int_0^{2\pi} K(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}m\omega_0^2 A^2\sin^2(\omega_0 t+\phi) = \frac{1}{4}m\omega_0^2 A^2$

c) $\frac{dE}{dt} = P = \vec{F}\cdot\vec{v} = -bmv^2$

$v^2 = \frac{2K(t)}{m} \Rightarrow \frac{dE}{dt} = -2bK(t) \Rightarrow \left(\frac{dE}{dt}\right)_{avg} = -2bK_{avg}$

d) $2K_{avg} \simeq E$ because $K_{avg} \simeq V_{avg} \simeq \frac{E}{2}$

$\Rightarrow \left(\frac{dE}{dt}\right)_{avg} = -bE$

e) The potential energy at the extrema $= \frac{1}{2}Kx_{max}^2$ = total Energy
where x_{max} decays like $e^{-bt/2}$

Therefore the energy decays like e^{-bt}

(4) a) $\Sigma F = ma$

$$F_{ext} + F_{spring} + F_{drag} = ma$$

$$F_0 \cos(\omega t) - K(x - x_{eq}) - cm\dddot{x} = m\ddot{x}$$

$$\ddot{x} + c\dddot{x} + \omega_0^2 (x - x_{eq}) - \frac{F_0}{m}\cos(\omega t) = 0 \quad \text{E of M}$$

where $\omega_0 = \sqrt{K/m}$

Assume $x(t) = x_{eq} + Re(\underline{A} e^{i\omega t})$

$$\dot{x}(t) = Re(i\omega \underline{A} e^{i\omega t})$$

$$\ddot{x}(t) = Re(-\omega^2 \underline{A} e^{i\omega t})$$

$$\dddot{x}(t) = Re(-i\omega^3 \underline{A} e^{i\omega t})$$

replace $F_0 \cos(\omega t)$ with $Re\left(\frac{F_0}{m} e^{i\omega t}\right)$

substituting and bring everything within a single $Re()$

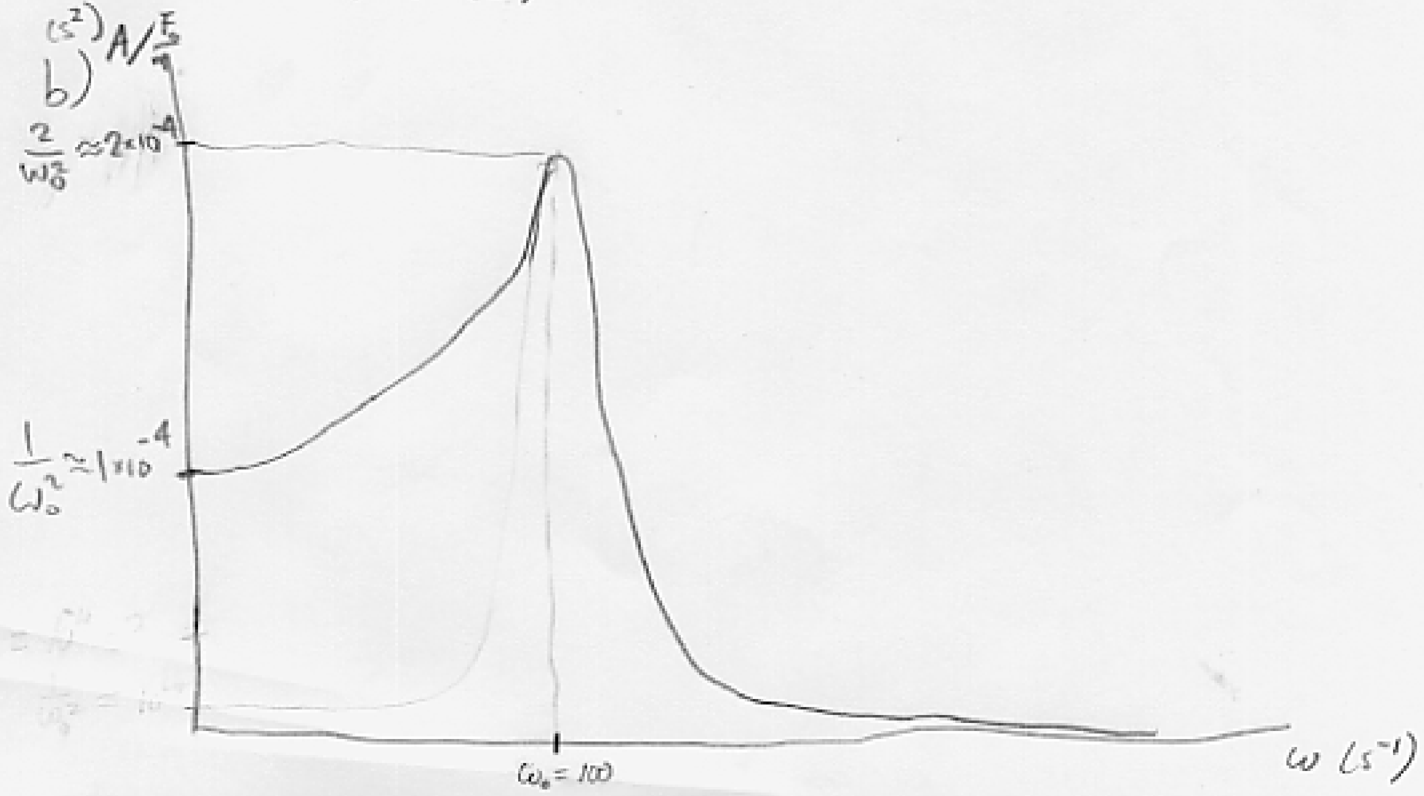
$$Re\left(-\omega^2 \underline{A} e^{i\omega t} - ci\omega^3 \underline{A} e^{i\omega t} + \omega_0^2 \underline{A} e^{i\omega t} - \frac{F_0}{m} e^{i\omega t}\right) = 0$$

$$\Rightarrow Re\left(e^{i\omega t}\left(\underline{A}(-\omega^2 - ci\omega^3 + \omega_0^2) - \frac{F_0}{m}\right)\right) = 0$$

$$\Rightarrow \underline{A}(-\omega^2 - ci\omega^3 + \omega_0^2) - \frac{F_0}{m} = 0 \Rightarrow \underline{A} = \frac{F_0/m}{\omega_0^2 - \omega^2 - ic\omega^3}$$

$$A = |\underline{A}| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (c\omega^3)^2}}$$

b)



5 a) $\lambda = \frac{c}{\nu} = \frac{3.0\times10^8 \frac{m}{s}}{9.00\times10^8 s^{-1}} = .33m$

$k = \frac{1}{\lambda} = \frac{1}{.33m} = 3.0m^{-1}$, $K = 2\pi k = 18.8 m^{-1}$

b) $\lambda = \frac{c}{f} = \frac{3.0\times10^8 \frac{m}{s}}{2.5\times10^9 s^{-1}} = .12m$

This wave length is smaller than most microwaves yet it is macroscopic. Therefore whereever there are nodes of the radiation there will be a cold spot noticable to humans.

c) $\lambda = 1.20m$ $v = 344 \frac{m}{s}$

$f = \frac{v}{\lambda} = \frac{344 \, m/s}{1.20 \, m} = 287 \, Hz$

$\lambda = .120m$ $v = 344 \, m/s$

$f = \frac{v}{\lambda} = \frac{344 \frac{m}{s}}{.120 \, m} = 2.87 \, kHz$

$\lambda = .0120m$ $v = 344 \frac{m}{s}$

$f = \frac{v}{\lambda} = \frac{344 \frac{m}{s}}{.0120 \, m} = 28.7 \, kHz$