Cornell University

Department of Physics

Phys 214 September 17, 2003

Waves, Optics, and Particles, Fall 2003

Homework Assignment # 4

(Due Thursday, September 25 at 5:00pm *sharp*.)

Agenda and readings for the week of September 22:

Skills to be mastered:

- distinguishing between speed of wave propagation and particle speed;
- understanding that the longitudinal component of tension τ_x is constant;
- getting τ_y knowing $\frac{\partial y}{\partial x}$;
- understanding that the "chunk" mass is $\mu\Delta x$;
- converting expressions like $\frac{\frac{\partial y}{\partial x}(x+\Delta x,t)-\frac{\partial y}{\partial x}(x,t)}{\Delta x}$ to the proper partial derivatives, $\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}(x,t)\right)$, in the limit $\Delta x \to 0$;
- interpretting "snapshots" $y(x, t = t_0)$ and "particle histories" $y(x = x_0, t)$;
- understanding the boundary conditions for both fixed and free ends of a string;
- sketching allowed modes for different types of boundary conditions;
- reading the wavelength off of pictures of different standing wave modes;
- extracting allowed frequencies from allowed wavelengths;
- confirming a given $y(x, t)$ as a solution to the wave equation;
- deriving the dispersion relation $\omega = f(k)$ from the standing wave form $y(x, t) = A\cos(\omega t)\cos(kx)$;
- relating the wave speed (v) to the physical parameters of the system in which the waves are propagating: i.e., $v = c \equiv \sqrt{\tau/\mu}$ for a string; $v = c \equiv \sqrt{B/\rho}$ for sound.

Lectures and Readings:

Readings marked YF are from the text Young and Freedman, *University Physics*, 10th edition. Readings marked LN are from the course lecture notes to be found at http://people.ccmr.cornell.edu/~muchomas/P214.

- Lec 8, 09/23 (Tue): Maxwell's equations in vacuum; Polarization Rule I for electromagnetic waves; differential forms for Maxwell's equations.
  Readings: LN "Other waves" Sec. 3–3.2.4; YF 33-1, 33-2, 33-3.
- Lec 9, 09/25 (Thu): "Let there be light"; Polarization Rule II for light.
  Readings: LN "Other waves" Sec. 3.2.5-4; YF 33-4.

Contents

1 Wave kinematics: particle speed versus wave speed

Do *Young & Freedman*, Problem 19-12 (p. 616).

2 Forces on a string

A string setup as in our physical realization for studying waves is plucked at its center. (See Figure 1, next page.) The mass of the string is $M = 10^{-2}$ kg, its length is $L = 1$ m, the applied tension is $\tau = 100$ N, and the string can sustain a maximum tension of $T = \sqrt{T_x^2 + T_y^2} = 200$ N before breaking.

(a) What is the maximum distance y (indicated in the figure) by which the string can be plucked before breaking?

(b) If the string is held in the plucked position with $y = 0.1$ m and then released, what will be the initial acceleration of each point of the string with $0 < x < L/2$? For each point with $L/2 < x < L$?
Hint: Use the wave equation derived in class.
Note: You may find it fun to think about the acceleration of the chunk at $x = L/2$ and what this all means in terms of how the string will move once released! We will answer that question later when we have the general solution to the wave equation.

3 Wave equation fundamentals

A standing wave on a string of mass m fixed at both ends ($x = 0$ and $x = L$) is described by

$$y(x,t) = A\sin(kx)\sin(\omega t)\ . \tag{1}$$

Express all answers below in terms of the fundamental quantities L, m, A, k, and ω.

(a) What is the x-component of the force due to the string on the fixed point $x = 0$? (Remember, the string is under tension so it pulls on whatever is holding it.)

(b) What is the y-component of the force due to the string on the fixed point $x = 0$ at any time t.

(c) Consider a tiny *chunk* of string of length dx between x and $x + dx$. Find the x- and y-components of the force on the left side of this chunk (at x) due to the rest of the string.

(d) Find the x- and y-components of the force on the right side of this chunk (at $x + dx$) due to the rest of the string.

(e) Find the net force on the chunk.

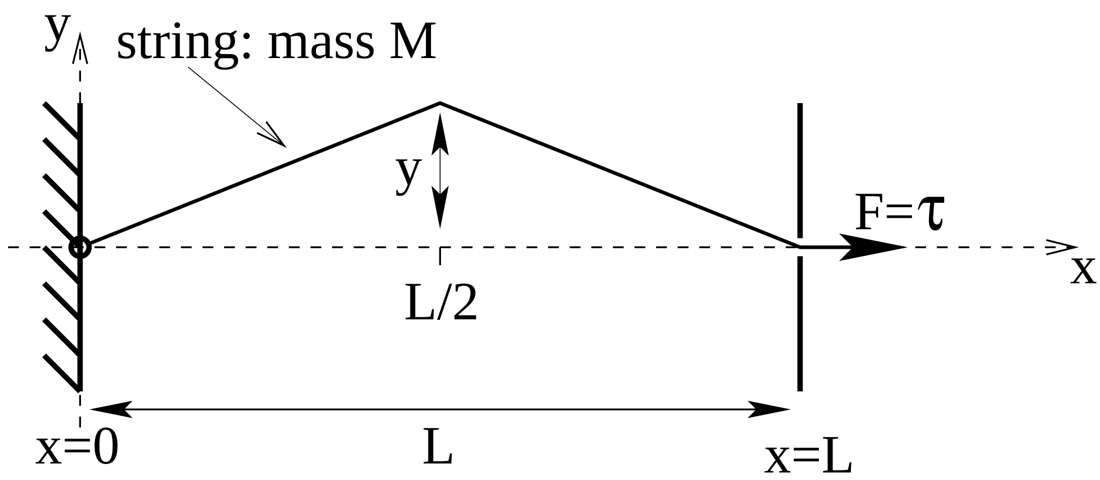

Figure 1: String plucked at its center to a distance y.

(f) Verify that $\sum \vec{F} = m\vec{a}$ works for the chunk in the limit $dx \to 0$. (Note that you should be able to do better than saying $0 = 0$.)

(g) The net force F_y on the chunk in the y-direction is proportional is to its displacement y from equilibrium. Use this to compute an effective spring constant $K = -F_y/y$. Compute the frequency you would expect from an object of mass equal to the mass of the chunk tied to a spring of constant K, and compare to ω.

4 Deriving your own wave equation

Take the same string system from class lecture and the lecture notes, but with drag on each chunk of the string contributing an addition force $\vec{F} = -bm_{\mathrm{ch}}\vec{v}$.

(a) Redraw the force-body diagram of Figure 2 from the lecture notes "Intro to Waves" including the new force. Explain (briefly, one or two short sentences) why this force does not affect motion in the x direction and hence does not modify Eq. (7) in the notes.

(b) Following the analysis of Section 4.2.2 of the lecture notes, rewrite in their new form any of Eq. (8), Eq. (11) and Eq. (13) which change due to the presence of the new force.
Hint: You should find in the end that your new equation of motion is equivalent to

$$\frac{\partial^2 y}{\partial t^2} + b\frac{\partial y}{\partial t} = c^2\frac{\partial^2 y}{\partial x^2},$$

where $c \equiv \sqrt{\tau/\mu}$.

(c) How quickly will the drag cause the amplitude of a standing wave to decay to $1/e$ of its original amplitude?
Hint: The answer is $1/a$ if the solution to the wave equation has the form

$$y(x,t) = Ae^{-at}\cos(\omega t)\cos(kx)$$

To find a, assume a solution with the complex representation

$$y(x,t) = \mathrm{Re}\left(Ae^{i\underline{\omega}t}e^{\pm ikx}\right)$$

where $\underline{\omega}$ is a complex frequency, solve for $\underline{\omega}$, and then take $a = \mathrm{Im}\ \underline{\omega}$.

5 Normal modes

(a) Do YF problem 19-15 (p. 616).
Note: In (c), they want you to find the combinations of A and λ which make the particle velocity achieve the conditions listed in the problem.

(b) Supposing that the apparatus in YF problem 19-15 is such that only waves with nodes at each end are allowed (closed boundary conditions), what is the length of the string if the situation with the 1.50 kg mass corresponds to the fundamental (longest wavelength) mode?

(c) For a string of the length you find in **(b)**, what values of the mass are needed to then give the first, second and third next modes?

(d) A simple model of a clarinet (a single-reed woodwind instrument) is a pipe of length 66.5 cm with one closed end (the reed end, a displacement node) and one open end (a displacement maximum). The speed of sound is 350 m/s. What are the frequencies of the lowest two modes within this simple model?
Just for fun: "Middle A" is 440 Hz and each "octave" represents a factor of two in frequency, so 110 Hz would be an A two octaves below middle A.

6 From the journal *Nature* to your problem set: Carbon nanotubes

Prof. McEuen in our Department of Physics studies the vibrations of tiny tubes (with nanometer diameter!) made entirely of carbon. The propagation of transverse waves along such tubes is described by the modified wave equation

$$\mu\frac{\partial^2 y}{\partial t^2} - \tau\frac{\partial^2 y}{\partial x^2} + F\frac{\partial^4 y}{\partial x^4} = 0 \qquad (2)$$

where μ and τ (as in lecture) are the the linear mass density and tension respectively, and F is an elastic parameter independent of the tension and characteristic of the nanotube.
Note: This modified equation also describes wires with stiffness (such as steel piano wires) rather than simple strings, something about which an astute student in the morning section asked.

(a) Show that the standing wave $y(x,t) = A\cos(\omega t)\cos(kx)$ is a solution to the nanotube wave equation (2), and derive the dispersion relation $\omega = f(k)$.

(b) Typical values for these tubes are $\mu = 3\times 10^{-15}$ kg/m, $\tau = 0.5\times 10^{-9}$ N, and $F = 4\times 10^{-26}$ N·m. Prof. McEuen's tubes are typically 1 μm$=10^{-6}$ m in length. Assuming a wavelength of 2 μm for the fundamental (lowest frequency) mode, how important is the correction term F? To answer this, compute the ratio $\omega/(ck)$ where c is the speed you would expect for a normal string; i.e., if the new, F term were not there. Given your results, for work good to a few percent, should Prof. McEuen consider the F term for the first few modes of his tubes?

(c) Below what wavelength (in Angstroms, 1 Å$=10^{-10}$m) does the new F-term change the frequency significantly? Specifically, determine the wavelength λ such that the frequency is twice as high as you would expect from the usual dispersion relation $\omega = ck$.