P214 Homework 4

1. YF Problem 19-12

(a) $y(x,t) = A\sin\left[\omega\left(t - \frac{x}{v}\right)\right] \stackrel{?}{=} -A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$

$\omega = kv \qquad k = \frac{2\pi}{\lambda}$

$\left[\omega\left(t - \frac{x}{v}\right)\right] = kvt - kx = \frac{2\pi}{\lambda}(vt - x)$

so $y(x,t) = A\sin\left[\frac{2\pi}{\lambda}(vt - x)\right] = -A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$

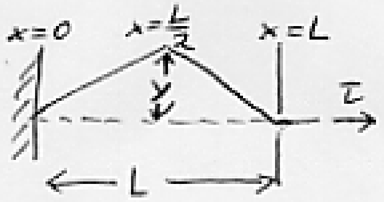
(b) $v_y = \frac{\partial y(x,t)}{\partial t} = -A\frac{2\pi v}{\lambda}\cos\left[\frac{2\pi}{\lambda}(x - vt)\right]$

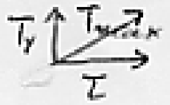
(c) maximum $|v_y|$ when $\cos[\dots] = 1$ $\quad |v_y| = v \quad A\frac{2\pi}{\lambda} = 1 \quad A = \frac{\lambda}{2\pi}$

for given v and λ, $|v_y|$ depends on A $\quad |v_y| < v \quad A\frac{2\pi}{\lambda} < 1 \quad A < \frac{\lambda}{2\pi}$

$|v_y| > v \quad A > \frac{\lambda}{2\pi}$

2.





(a) $\frac{dy}{dx} = \frac{T_y}{\tau} = \frac{\sqrt{T_{max}^2 - \tau^2}}{\tau} = \frac{\sqrt{200^2 - 100^2}}{100} = \sqrt{3}$

$\frac{y}{L/2} = \frac{dy}{dx} \qquad y = \sqrt{3}\cdot 0.5\,m = 0.87\,m$

(b) Initial acceleration with $y = 0.1\,m$

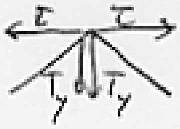
Wave equation $\left.\frac{\partial^2 y}{\partial t^2}\right|_{t=0} = c^2\frac{\partial^2 y}{\partial x^2} = c^2\frac{\partial}{\partial x}\frac{\partial y(x,0)}{\partial x}$

For $0 < x < \frac{L}{2}$ and $\frac{L}{2} < x < L$, the slope is constant, namely $\pm\frac{0.1}{0.5}$ respectively

The initial acceleration of each point will be <u>zero</u>.

You can also look at the forces on a small chunk, and find that ΣF is zero for $0 < x < \frac{L}{2}$ and $\frac{L}{2} < x < L$, again because the slope is constant.

At $x = \frac{L}{2}$, the net force is <u>not</u> zero, $F_{net} = 2T_y$ (down)



For a chunk of mass $\mu \Delta x$

$$\mu \Delta x \, a_y = 2T_y$$
$$a_y = \frac{2T_y}{\mu \Delta x}$$

As $\Delta x \to 0$ $\quad a_y \to \infty$

3. Wave Equation Fundamentals

$$y(x,t) = A \sin(kx) \sin(\omega t)$$

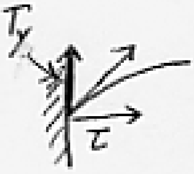
describes a standing wave on a string of mass m and length L, fixed at both ends.
Given A, k, ω, L, m.

(a) Force on fixed point at $x = 0$, x-component

$$F_x = \tau \qquad c = \sqrt{\tau/(m/L)} \quad \text{and } c = \frac{\omega}{k}$$

$$F_x = \tau = \frac{\omega^2}{k^2} \frac{m}{L}$$

(b)

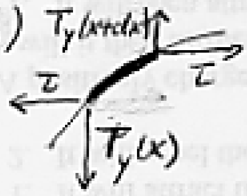


tension is tangential to string

$$\frac{T_y}{\tau} = \frac{\partial y(0,t)}{\partial x} = Ak \cos kx \sin \omega t \Big|_{x=0}$$
$$= Ak \sin \omega t$$

$$F_y = T_y = \tau A k \sin \omega t = \frac{\omega^2}{k} \frac{m}{L} A \sin \omega t$$

(c)



left side $F_x(x) = -\tau$

$$F_y(x) = -T_y(x) = -\tau \frac{\partial y(x,t)}{\partial x} = -A \frac{\omega^2}{k} \frac{m}{L} \cos kx \sin \omega t$$

(d) right side $F_x(x+dx) = +\tau$

$$F_y(x+dx) = T_y(x+dx) = T_y(x) + \frac{\partial T_y}{\partial x} dx$$

3 (d) cont.d ③

$$F_y(x+dx) = T_y(x) - A\omega^2 \frac{m}{L} \sin(kx)\sin(\omega t)\, dx$$

$$= A\frac{\omega^2}{k}\frac{m}{L}\left[\cos(kx)\sin(\omega t) - k\sin(kx)\sin(\omega t)\, dx\right]$$

(e) Net force

$$F_{net,y} = F_y(x+dx) - F_y(x) = -\frac{A\omega^2 m}{L}\sin(kx)\sin(\omega t)\, dx$$

$$= -A\omega^2\mu \sin(kx)\sin(\omega t)\, dx$$

$$F_{net,x} = 0$$

(f) $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{F_{net}}{dm} = \dfrac{F_{net}}{\mu\, dx} = -A\omega^2 \sin(kx)\sin(\omega t)$

$dm = \mu\, dx$
$= \frac{m}{L} dx$
$=$ mass of chunk
$= m_{chunk}$

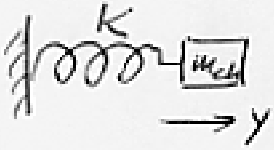
taking derivative of $y(x,t)$

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -A\omega^2 \sin(kx)\sin(\omega t)$$

(g) $\vec{F}_{net} = -K\vec{y}$ (K = spring constant)

find effective K:

$$K = \frac{F_{net}}{y} = \frac{A\omega^2\mu\sin(kx)\sin(\omega t)\, dx}{A\sin(kx)\sin(\omega t)} = \omega^2 m \frac{dx}{L} = m_{chunk}\,\omega^2$$

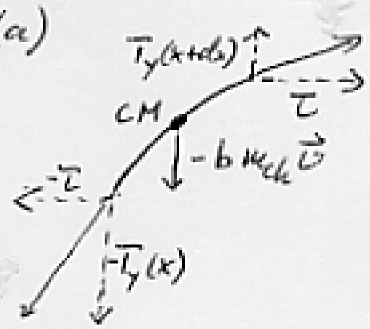


$F_{spring} = Ky = -m_{ch}\dfrac{d^2 y}{dt^2}$

find $\omega^2 = \dfrac{K}{m_{chunk}}$ through usual analysis, the same ω^2 as above.

4. Our own wave equation, with drag $\vec{F}_d = -b m_{ch} \vec{v}$ (4)

(a)



assume that $\vec{v}$ is upward at the time t.

Since there is no motion in the x-direction, i.e. string particles only undergo SHM in the y-direction, there is no x-component of $\vec{F}_d$, only drag in y-direction.

(b) $m\vec{a} = \Sigma \vec{F}$

x-direction: $m_{ch} a_x = \tau - \tau = 0$

y-direction $m_{ch} a_y = T_y(x+dx) - T_y(x) - b m_{ch} \vec{v}$

$$\mu dx \frac{\partial^2 y}{\partial t^2} = T_y(x+dx) - T_y(x) - b\mu dx \frac{\partial y}{\partial t}$$

divide by dx, and let $dx \to 0$

$$\mu \frac{\partial^2 y}{\partial t^2} + b\mu \frac{\partial y}{\partial t} = \frac{\partial T_y}{\partial x}$$

from geometry, i.e. overall tension parallel to string:

$$\frac{T_y}{T_x} = \frac{T_y}{\tau} = \frac{\partial y}{\partial x} \qquad T_y = \frac{\partial y}{\partial x} \tau$$

$$\mu \frac{\partial^2 y}{\partial t^2} + b\mu \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \tau$$

$$\boxed{\frac{\partial^2 y}{\partial t^2} + b \frac{\partial y}{\partial t} = \frac{\tau}{\mu} \frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial x^2}}$$

with $c^2 = \frac{\tau}{\mu}$

5 Normal Modes

(a) YF 19-15 (it's almost like the lab)

(a)

$\mu = 0.0550$ kg/m

$$c = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{mg}{\mu}} = \sqrt{\frac{1.5 \times 9.8}{0.055}} \text{ m/s} = 16.35 \text{ m/s}$$

(b) $\lambda = \frac{c}{f} = \frac{16.35 \text{ m/s}}{120/\text{s}} = 0.136 \text{ m}$

(c) increase mass to 3 kg, doubles tension

(a)' : $c' = \sqrt{2} \cdot 16.35 \text{ m/s} = 23.1 \text{ m/s}$

(b)' : $\lambda' = \sqrt{2} \times 0.136 \text{ m} = 0.193 \text{ m}$ (remember f is unchanged)

(b) standing wave in fundamental mode, $n=1$

$$L = \frac{\lambda_1}{2} \left(= \frac{c}{2f}\right) = 0.068 \text{ m} = 6.8 \text{ cm}$$

(c) to get higher modes with fixed f and L, change c, i.e. change the tension by changing the suspended mass.

$n = 1 \quad \lambda_1 = 2L \quad c_1 = \lambda_1 f = 2Lf \quad m_1 = 1.5$ kg

$n = 2 \quad \lambda_2 = L \quad c_2 = Lf \quad m_2 = \frac{c_2^2 \mu}{g} = \frac{L^2 f^2 \mu}{g} = 0.37$ kg

$n = 3 \quad \lambda_3 = \frac{2L}{3} \quad c_3 = \frac{2}{3} L f \quad m_3 = \frac{4L^2 f^2 \mu}{9g} = 0.167$ kg

$\vdots$

$c_n = \frac{2L}{n} f \quad m_4 = \frac{L^2 f^2 \mu}{4g} = 0.094$ kg

Since $c = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{mg}{\mu}}$

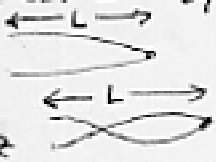
$$m = \frac{c^2 \mu}{g} = \frac{\lambda_n^2 f^2 \mu}{g}$$

note that $m_n = \frac{m_1}{n^2}$

as $\lambda_n = \frac{2L}{n} = \frac{\lambda_1}{n}$

(d) clarinet, $L = 65$ cm

pressure antinode at mouthpiece, pressure node at bell (open end).



$L = \frac{\lambda_1}{4} \quad f_1 = \frac{v_{sound}}{\lambda_1} = \frac{350 \text{ m/s}}{4 \cdot 0.65 \text{ m}} = 134.6 \text{ Hz}$

$L = \frac{3\lambda_2}{4} \quad f_2 = \frac{v_{sound}}{\lambda_2} = \frac{3 \cdot 350 \text{ m/s}}{4 \cdot 0.65 \text{ m}} = 3f_1 = 403.8 \text{ Hz}$

Due to the presence of the mouthpiece, and the width of the pipe, the instrument is effetively longer, playing D^b at 147 Hz.

6. Nanotubes

$$\mu\frac{\partial^2 y}{\partial t^2} - \tau\frac{\partial^2 y}{\partial x^2} + F\frac{\partial^4 y}{\partial x^4} = 0 \qquad \underline{\text{transverse}}\text{ waves}$$

(a) $y = A\cos\omega t\cos kx$ is a solution, if $\omega = f(k)$ as shown

$$\frac{\partial^2 y}{\partial t^2} = -A\omega^2 y$$

$$\frac{\partial^2 y}{\partial x^2} = -Ak^2 y \qquad -\mu \cancel{A}\omega^2 + \tau \cancel{A} k^2 + F\cancel{A}k^4 = 0$$

$$\frac{\partial^4 y}{\partial x^4} = +Ak^4 y \qquad \omega^2 = \frac{\tau}{\mu}k^2 + \frac{F}{\mu}k^4$$

$$= \frac{\tau}{\mu}k^2\left(1 + \frac{F}{\tau}k^2\right)$$

(b) $\mu = 3\cdot 10^{-15}$ kg/m

$\tau = 0.5\times 10^{-9}$ N

$F = 4\cdot 10^{-26}$ N·m

$L = 1\times 10^{-6}$ m

lowest mode $\lambda = 2\cdot 10^{-6}$ m

for flexible string, $c = \frac{\omega}{k} = \left(\frac{\lambda}{T}\right)$, i.e. $\frac{\omega}{ck} = 1$

$$(ck)_{flex} = \omega_{flex} = 2\pi\frac{c}{\lambda} = 2\pi\frac{\sqrt{\tau/\mu}}{\lambda} = 2\pi\frac{\sqrt{0.5\cdot 10^{-9}/3\cdot 10^{-15}}}{2\cdot 10^{-6}} = 1.28\cdot 10^{9}\,s^{-1}$$

$$\omega = \left[c^2\frac{4\pi^2}{\lambda^2}\left(1 + \frac{F}{\tau}\frac{4\pi^2}{\lambda^2}\right)\right]^{1/2}$$

$$= (ck)_{flex}\left(1 + \frac{F}{\tau}\frac{4\pi^2}{\lambda^2}\right)^{1/2}$$

$$\frac{\omega}{(ck)_{flex}} = \left(1 + \frac{F}{\tau}\frac{4\pi^2}{\lambda^2}\right)^{1/2} = \left(1 + 7.9\cdot 10^{-4}\right)^{1/2} = 0.000395 + 1$$

0.04 %

(c) For what wavelength is $\frac{\omega}{ck} = 2$?

$$2 = \left(1 + \frac{F}{\tau}\frac{4\pi^2}{\lambda^2}\right)^{1/2}$$

$$3 = \frac{F}{\tau}\frac{4\pi^2}{\lambda^2}$$

$$\lambda^2 = \frac{4\pi^2 F}{3\tau} = 1.05\cdot 10^{-15}\,m^2 \qquad \lambda = 3.24\cdot 10^{-8}\,m = 324\,\text{Å}$$

this is about 60 times less than the wavelength of the fundamental