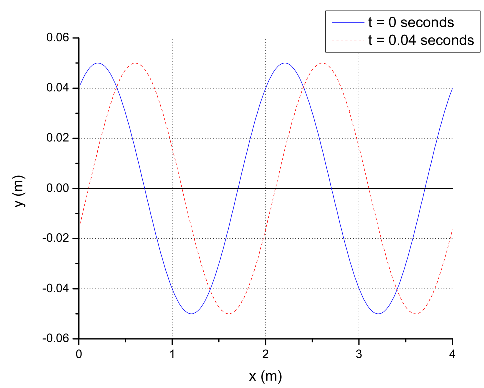
Physics 214 Fall 2003
Homework Assignment #5 Solutions
by Lisa Larrimore

1 Particle histories versus snapshots

(a) & (b) These snapshots will be easier to sketch if we rewrite the solution for the wave as

$$y(x,t) = \Re\{\underline{A}e^{i(kx-\omega_0 t)}\} = |\underline{A}|\cos(kx - \omega_0 t - \phi). \tag{1}$$

Since $\underline{A} = (4-3i)\times 10^{-2}$ m, we know that $|\underline{A}| = 0.05$ m and $\phi = -\tan^{-1}\frac{3}{4}$. We are also given that $k = 3.14$ m^{-1} and $\omega_0 = 31.4$ s^{-1}. To plot y at fixed t, we just fix the t in Eq. (1) to 0 s for (a) and 0.04 s for (b), giving the graphs shown.

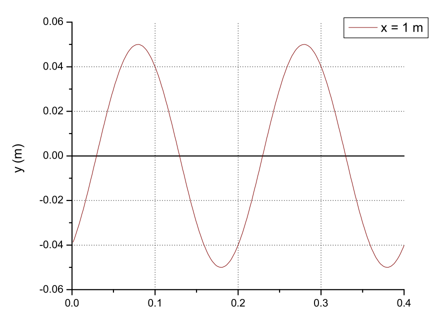


To estimate the speed of the wave, we notice that in 0.04 seconds, the first maximum has traveled about 0.4 m (from 0.2 m to 0.6 m), and thus

$$v = \frac{\text{distance}}{\text{time}} = \frac{0.4 \text{ m}}{0.04 \text{ s}} = 10 \text{ m/s}. \tag{2}$$

This is the same result obtained using $v = \omega_0/k$.

(c) To sketch a particle history, $y(t)$ at fixed x, we just fix x in Eq. (1) to be 1 meter, giving us the graph shown.



2 Frequency of spring waves

(a) The lowest frequency standing wave *for boundary conditions with both ends fixed* is the one with $\lambda = 2L$. Using this and the fact that $\tau = K(L - L_0)$, we can express the angular frequency as

$$\omega = kc = \frac{2\pi}{\lambda}c = \frac{2\pi}{2L}\sqrt{\frac{\tau}{\mu}} = \frac{\pi}{L}\sqrt{\frac{K(L-L_0)}{M/L}} = \pi\sqrt{\frac{K(L-L_0)}{ML}}. \tag{3}$$

(b) We can now take the limit as $L \gg L_0$:

$$\lim_{L \gg L_0} \omega = \pi\sqrt{\frac{KL}{ML}} = \pi\sqrt{\frac{K}{M}}, \tag{4}$$

which is independent of the length L.

3 Damped boundary conditions and normal modes

(a) We want to apply

$$\sum \vec{F}^{(ext)} = m\vec{a}^{(\text{c of m})} \tag{5}$$

to the ring at $x = 0$ in Figure (1) on the problem set. Since the ring has zero mass, the right hand side is zero, so the sum of the external forces must be zero. Applying this to the x-component of the external forces simply tells us that $N = \tau$. The y-component, however, gives us

$$\sum F_y = -b\frac{\partial y(x=0,t)}{\partial t} + \tau\frac{\partial y(x=0,t)}{\partial x} = 0, \tag{6}$$

resulting in the relation

$$b\frac{\partial y(x=0,t)}{\partial t} = \tau\frac{\partial y(x=0,t)}{\partial x}. \tag{7}$$

(b) In the limit $b/\tau \to 0$, Eq. (7) becomes $\frac{\partial y(x=0,t)}{\partial x} = 0$, which is the condition for a free boundary condition. Since the other side of the string also has a free boundary condition, the allowed wave vectors for normal modes are $k = \frac{n\pi}{L}$.

(c) In the limit $b/\tau \to \infty$, Eq. (7) becomes $\frac{\partial y(x=0,t)}{\partial t} = 0$, which is the condition for a fixed boundary condition. The allowed wave vectors are $k = \frac{2\pi}{L}\left(\frac{n}{2} - \frac{1}{4}\right)$, as determined in lecture.

(d) The standing wave $y(x,t) = A_0 \sin(kx + \phi_0)\cos(\omega t)$ cannot satisfy Eq. (7) for any value of b other than 0 or ∞. The left hand side of Eq. (7) at fixed x would be proportional to $\sin(\omega t)$, and the right hand side would be proportional to $\cos(\omega t)$, so the two sides could not be equal for all t unless one side is equal to zero.

We can understand this by considering conservation of energy. The drag force $\vec{F} = -b\vec{v}$ is a non-conservative force, which means that energy is lost (to heat, etc.) as it acts. If energy is lost, the amplitude of oscillation will decay away to zero, which means that the expression for $y(x,t)$ does not describe the wave for all t, and it is therefore not a solution to the equation of motion. The general standing wave can only satisfy the equation of motion if energy is *not* lost to the drag force, which means that either b has to be zero or $\vec{v}$ has to be zero (b has to be infinite).

4 Sound waves

(a) Y&F 19-19: For this problem, we just need to plug numbers into the formula $B = \rho_0 c^2$.

$$\text{19a) } B = \rho_0 c^2 = \rho_0(\lambda f)^2 = (1300 \text{ kg/m}^3)[(8 \text{ m})(400 \text{ Hz})]^2 = 1.3 \times 10^{10} \text{ Pa.} \tag{8}$$

$$\text{19b) } B = \rho_0 c^2 = \rho_0(L/t)^2 = (6400 \text{ kg/m}^3)[(1.5 \text{ m})/(3.9 \times 10^{-4} \text{ s})]^2 = 9.4 \times 10^{10} \text{ Pa.} \tag{9}$$

(b) Equation 11 from Section III of the class notes tell us that

$$P(x) = P_0 - B\frac{\partial s}{\partial x}. \tag{10}$$

We want $P(x)$ to change by $\pm 0.1P_0$, which means $P(x)$ oscillates between $1.1P_0$ and $0.9P_0$. Under these conditions, Eq. (10) becomes $\pm 0.1P_0 = -B\frac{\partial s}{\partial x}$. B can be easily calculated as in part (a):

$$B = \rho_0 c^2 = (1.3 \text{ kg/m}^3)(344 \text{ m/s})^2 = 1.5 \times 10^5 \text{ Pa.} \tag{11}$$

We also need $\frac{\partial s}{\partial x}$:

$$\frac{\partial s}{\partial x} = A_0 k \cos(kx)\cos(\omega t), \tag{12}$$

which has maximum and minimum values at $\pm A_0 k$. The wave vector k is

$$k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = \frac{2\pi \times 440 \text{ Hz}}{344 \text{ m/s}} = 8.04 \text{ m}^{-1}. \tag{13}$$

We are also given that $P_0 = 1.01 \times 10^5$ Pa. We can now plug all these numbers in to find the amplitude at which $\pm 0.1P_0 = -B\frac{\partial s}{\partial x}$:

$$A_0 = \frac{0.1P_0}{Bk} = \frac{0.1(1.01 \times 10^5 \text{ Pa})}{(1.5 \times 10^5 \text{ Pa})(8.04 \text{ m}^{-1})} = 0.8 \text{ cm.} \tag{14}$$

5 Prelim Practice: Standing Waves in Sound Tubes

The resonant frequency depends on both the size of the tube (you can play higher notes on a little piccolo than on a flute) and on the density of the air inside (having your larynx full of a light gas like helium makes your voice higher than when it is full of the air that is usually around us). (Resonant frequency also depends on bulk modulus B, but that is fixed for these tubes.) For a tube of length L with both ends closed, the lowest frequency corresponds to $\lambda = 2L$. We can thus write the lowest resonant frequency as

$$f = \frac{v}{\lambda} = \frac{1}{\lambda}\sqrt{\frac{B}{\rho}} = \frac{1}{2L}\sqrt{\frac{B}{\rho}}. \tag{15}$$

Note that the dependence on L and ρ is not the same – if you halve the length of the tube, you need to quadruple the air density in order to keep the same resonant frequency. This is exactly what is changed between tubes A and C, so that is the correct answer. Note that B will have a lower frequency and D will have a higher frequency than both A and C.

6 Prelim Practice: Sound Waves versus Waves on a String

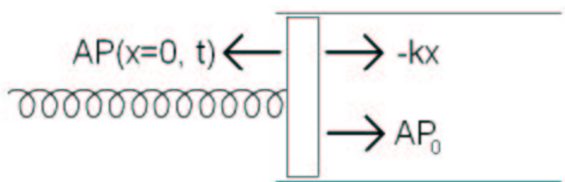
We first need to remember the similarities between string waves and sound waves. The vertical displacement y of the string corresponds to the horizontal displacement s of a chunk of air in a sound wave. The mass density μ of a string corresponds to the mass density ρ_0 of the air (or other medium) in which the sound is propagating. Finally, the tension τ on the string corresponds to the bulk modulus B; they both describe the "stiffness" of the system.

Now, we simply make these substitutions to turn the conservation of energy equation for strings into one for sound:

$$-\frac{\partial}{\partial x}\left[B\frac{\partial s}{\partial x}\frac{\partial s}{\partial t}\right]+\frac{\partial}{\partial t}\left[\frac{1}{2}B\left(\frac{\partial s}{\partial t}\right)^2+\frac{1}{2}\rho_0\left(\frac{\partial s}{\partial t}\right)^2\right]=0. \tag{16}$$

7 Prelim Practice: Generalized Boundary Conditions for a Sound Wave

(a) We want to determine all the forces acting on the piston when it is at position x. There are no long-range forces (except for gravity, which is in the y-direction), so we only need to consider contact forces. We have a force from the spring equal to $-k(x-x_{eq})=-kx$ (since we have set x_{eq} to be our origin). We also have the pressure of the air on both sides pushing on the piston. The air outside has pressure P_0, and the air inside has pressure $P(x,t)$. Since we are considering small amplitude waves and are allowed to ignore the length of the tube, we can approximate $P(x,t)\approx P(x=0,t)$. The free body diagram from these forces is then:



(b) Since the piston is massless, Newton's Second Law tells us that the sum of the external forces on the piston must be zero:

$$0=\sum F_x=-kx+A\left[P_0-P(x=0,t)\right] \tag{17}$$

We are not allowed to have $P(x=0,t)$ in our final solution, so we rewrite it using the constitutive relation for sound,

$$P(x,t)=P_0-B\frac{\partial s(x,t)}{\partial x}. \tag{18}$$

Substituting this into Eq. (17) gives our equation of motion:

$$kx=AB\frac{\partial s(0,t)}{\partial x}. \tag{19}$$