Cornell University

Department of Physics

Phys 214 October 22, 2003

Waves, Optics, and Particles, Fall 2003

Homework Assignment # 8

(Due Thursday, October 30 at 5:00pm *sharp*.)

Agenda and readings for the week of October 27:

Skills to be mastered:

- Be able to use complex wave amplitudes;
- Understand the relation between wave intensity and its complex amplitude;
- Understand the superposition principle for complex wave amplitudes;
- Understand the derivation of the interference pattern from two or more narrow slits;
- Be able to determine the properties of a slit system that produces a known interference pattern;
- Be able to use phasors to illustrate interference.

Lectures and Readings:

Readings marked YF are from the text Young and Freedman, *University Physics*, 10th edition. Readings marked LN are from the course lecture notes to be found at http://people.ccmr.cornell.edu/~muchomas/P214.

- Lec 17, 10/28 (Tue): N-slit interference pattern
  Readings: LN "Wave Phenomena II: Interference," Sec. 3.3.
- Lec 18, 10/30 (Thu): Finite-slit interference
  Readings: LN "Wave Phenomena II: Interference," Sec. 4; YF 38-3.

Contents

1 Transmission and Reflection in Strings (I)

In this problem, you will consider transmission and reflection for waves on a string. In this case, a point mass attached to the string at $x = 0$ creates the disruption to normal propagation (Figure 1). The string has mass per unit length μ, applied tension τ, and may be regarded for this problem as infinitely long. The mass has mass m.

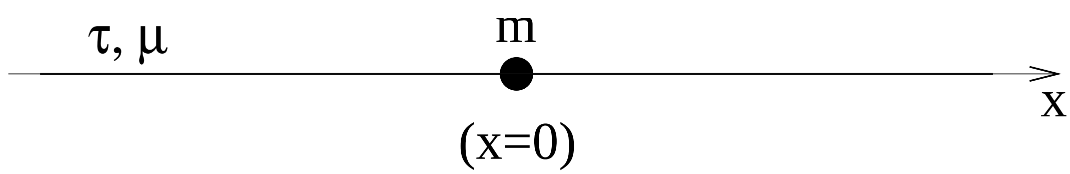

Figure 1: Point mass attached to string at position $x = 0$.

1.1 Complex solutions to the wave equation

Explain why, for a sinusoidal pulse incoming from the left and moving to the right, the solution to the wave equation has the form

$$\begin{aligned} y_0(x \le 0, t) &= \mathrm{Re}\left[\underline{A}e^{-i\omega t}\left(e^{ikx} + \underline{R}e^{-ikx}\right)\right] \\ y_1(x \ge 0, t) &= \mathrm{Re}\left[\underline{A}e^{-i\omega t}\left(\underline{T}e^{ikx}\right)\right]. \end{aligned} \tag{1}$$

Your explanation can be simply the identification of the meaning of each term in each equation.

1.2 Boundary conditions

Explain why the following two boundary conditions must hold at the point $x = 0$,

$$y_0(x = 0, t) = y_1(x = 0, t) \tag{2}$$

$$\tau \left.\frac{\partial y_1(x,t)}{\partial x}\right|_{x=0} - \tau \left.\frac{\partial y_0(x,t)}{\partial x}\right|_{x=0} = m \left.\frac{\partial^2 y_1(x=0,t)}{\partial t^2}\right|_{x=0}. \tag{3}$$

1.3 Transmission and Reflection coefficients

Combine Eqs. (1-3) to find two simplified equations for the unknowns $\underline{R}$, $\underline{T}$.
Hint: One of your simplified equations should be $1 + \underline{R} = \underline{T}$.

Solve your equations to find $\underline{R}$ and $\underline{T}$. (Don't worry if your solutions contain factors of i.)

What is the ratio of the amplitude (the actual (1/2)(max-min) "amplitude", not the "complex amplitude") of the transmitted wave to the real amplitude of the incoming wave?

What is the ratio of the real amplitude of the reflected wave to that of the incoming wave?

The energy of each wave is proportional to the square of its amplitude. Use this fact to verify that energy is conserved.

2 Transmission and Reflection in Strings (II)

A thin string (mass per unit length μ_0, wave speed c_0) of length a is attached to a wall at $x = 0$. The other end of the string is attached to a thicker string (mass per unit length μ_1, wave speed c_1), see Fig. 2.

A sinusoidal wave of amplitude A and wavevector k_1 travels in from the right along the thick string and is then reflected. Using complex representation, the motion of the string can be described by

$$\begin{aligned} y_0(x<a,t) &= \Re[e^{-i\omega t}(\underline{B}e^{-ik_0x}+\underline{C}e^{ik_0x})], \\ y_1(x>a,t) &= \Re[\underline{A}e^{-i\omega t}(e^{-ik_1(x-a)}+\underline{r}e^{ik_1(x-a)})]. \end{aligned} \tag{4}$$

where $\underline{r}$ is the reflection coefficient. In this problem, you will find $\underline{r}$ in two different ways.

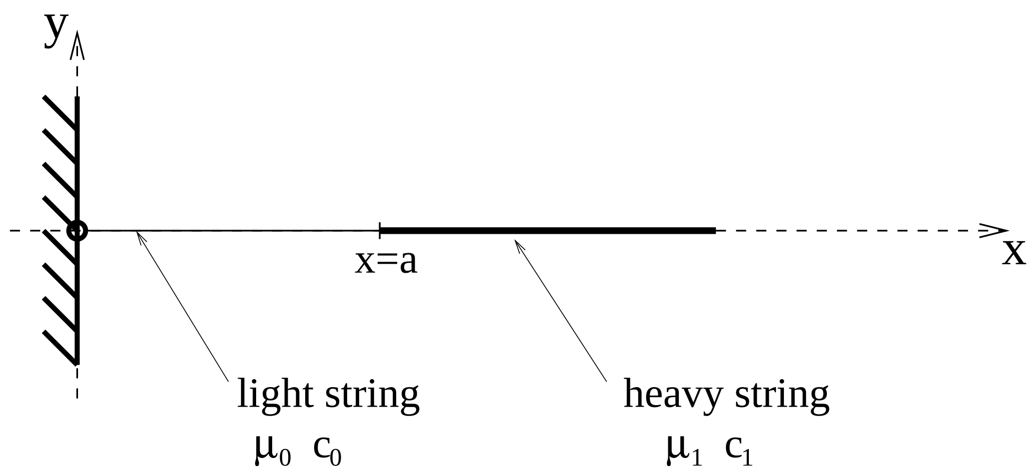

Figure 2: Transmission and reflection in strings.

2.1 Matching at the Boundary

The first way involves using boundary conditions at $x=0$ and $x=a$:

(a) Using the boundary condition at $x=0$, show that the motion of the light string is described by

$$y_0(x<a,t) = \Re[e^{-i\omega t}\underline{D}\sin(k_0x)]. \tag{5}$$

How is the coefficient $\underline{D}$ related to $\underline{B}$ and $\underline{C}$?

(b) Write down the equations that $y_0(x,t)$ and $y_1(x,t)$ should satisfy at $x=a$. What is the physical meaning of each equation?

(c) Solve these equations to obtain r. Express your answer in terms of c_0, c_1, k_1, and a.

2.2 Sums over Histories

The second way is to use the "sum over histories" idea developed in the lecture notes. The reflection and transmission amplitudes at the interface of two *half-infinite* strings can be obtained by analogy with sound waves. For example, for waves going from the light string to the heavy string we have

$$\begin{aligned} R_{01} &= \frac{z_0-z_1}{z_0+z_1}; \\ T_{01} &= \frac{2z_0}{z_0+z_1}, \end{aligned} \tag{6}$$

where the impedance for strings is $z=\mu c$.

(a) What are the transmission and reflection coefficients for waves going from the heavy string to the light string, R_{10} and T_{10}?

(b) What is the complex amplitude (at $x = a$) of the wave that was reflected from the interface between the strings *without* entering the light string? What about the wave that passed into the light string, was reflected from the boundary, and then passed back into the heavy string?

(c) *Challenge problem!* Generalizing your result in part (b), obtain the full complex amplitude of the right-moving wave in the heavy string at $x = a$ by summing over all possible back-and-forth reflections within the light part of the string. Use this result to find r.
HINT: To simplify your expression, use the formula for the sum of a geometric series (which also works for complex numbers): $\underline{a} + \underline{a}\,\underline{x} + \underline{a}\,\underline{x}^2 + ... = \underline{a}/(1 - \underline{x})$.

3 The Two Towers

Two radio towers, A and B, located at a distance d apart from one another transmit radio waves of wavelength λ. The emitted waves have exactly the same phase. The waves are observed by a receiver which is located far away from the towers, see Fig. 3. When the transmitter in the tower B is turned off, the intensity of the received signal is I_A; when the transmitter in the tower A is turned off, the intensity becomes I_B. Now, consider the situation when both transmitters are on.

(a) Find the expression for the intensity of the received signal I as a function of the angle θ (Simplify your expression so that the final answer involves no complex numbers!).

(b) What is the maximal value of the intensity I? At which values of θ do the maxima occur?

(c) What is the minimal value of the intensity I? At which values of θ do the minima occur?

(d) If the waves emitted by the towers were *out of phase*, would the minimal/maximal values of I change? What about the angles at which the minima/maxima occur?

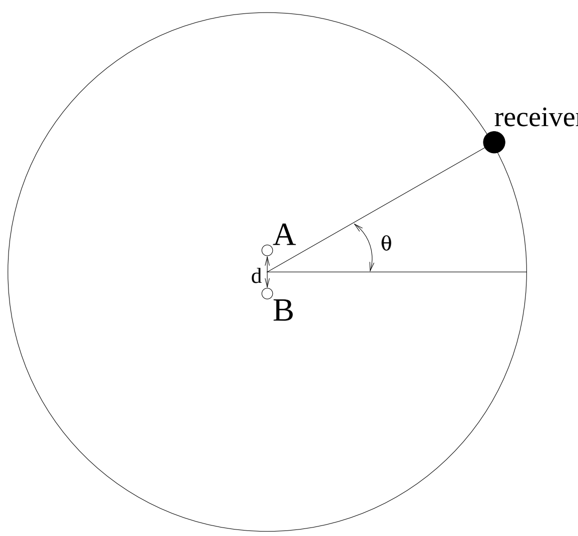

Figure 3: The two towers.

4 Phasors

Phasors provide a convenient way to analyze interference phenomena. A phasor is simply a two-dimensional vector that represents a complex number: the x-component of a phasor represents the real ($\Re\mathfrak{e}$) part of a complex number and the y-component represents the imaginary ($\Im\mathfrak{m}$) part. The polar representation of complex numbers also has a neat geometric interpretation with phasors: the angle that the phasor makes with the x-axis represents the phase of the complex number, and the length of the phasor represents its magnitude. To represent a *sum* of two or more complex numbers, you can now simply add the corresponding phasors, using the usual rules of vector addition.

Example: the sum $2 + 4e^{i\pi/3} \approx 5.3e^{i0.714\pi}$ can be represented by the phasor diagram in Fig. 4. We

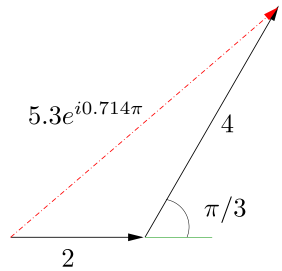

Figure 4: Adding complex numbers with phasors.

can use phasors to illustrate the sum inside the absolute value bars in Eq. (8) (p. 7 of the Class Notes "Wave Phenomena II: Interference".) Each phasor represents the complex amplitude of the light coming from one source. When we sum these phasors, the resultant phasor represents the complex amplitude of the superposition of light coming from all of the sources.

Example: A minimum for three equivalent narrow slits can result from either of the phasor sums in Fig. 5.

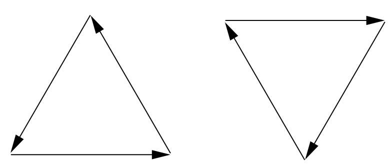

Figure 5: Phasor diagram for 3-slit interference.

Draw the phasor diagrams to illustrate the following cases of interference. Label magnitudes and angles. If there is more than one possibility (as in the example above), illustrate all of them.

(a) Principal maximum for 5 narrow slits;

(b) Minimum for 5 narrow slits ;

(c) Secondary maximum for 4 narrow slits (don't worry about making the angles exact—an approximation is fine);

(d) Minimum for 2 narrow slits where one is three times as wide as the other ($I_1 = 3I_2$).

5 Using Interference to Study Stars

Two telescopes, placed at a distance D from one another, receive light of wavelength λ from a distant star. The star is at an angle θ above horizon. Light from both telescopes is then transmitted to the sensor S which is located exactly halfway between them (see Fig. 6).

(a) What is the difference in the distance traveled by the light received by telescope A and telescope B?

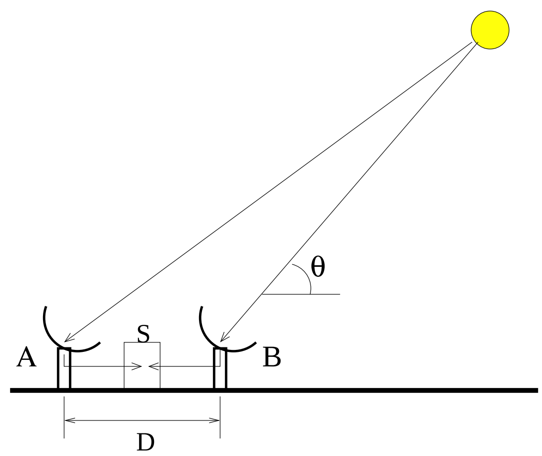

Figure 6: Stellar Interference.

(b) What is the intensity of light measured by the sensor? Express your answer in terms of the angle θ, distance D, wavelength λ, and the intensity I_0 that would be measured if only one of the telescopes was used.

(c) What are the values of $\cos\theta$ for which the intensity is at a maximum? At a minimum?

(d) If the star is moving, θ will change with time. By how much does $\cos\theta$ have to change to cause a large variation in the light intensity (e.g., from a maximum to the neighboring minimum)?

(e) In a typical situation, $D \gg \lambda$ (typical numbers would be $D \approx 1$ m, $\lambda \approx 10^{-6}$ m). Explain how the setup described in this problem can be used to detect tiny shifts in star positions, unobservable by naked eye.