1. a) Sinusoidal pulse, incoming from left and moving to the right.

$$y_0(x \le 0, t) = \mathrm{Re}\left[Ae^{-i\omega t}\left(e^{ikx} + Re^{-ikx}\right)\right]$$

$$= \mathrm{Re}\left[Ae^{i(kx-\omega t)} + RAe^{-i(kx+\omega t)}\right]$$

Incident wave moving in $+x$ direction, amplitude A

Reflected wave (from $x=0$) moving in $-x$ direction amplitude RA implicitly defining R

$$y(x \ge 0, t) = \mathrm{Re}\left[Ae^{-i\omega t}\left(Te^{ikx}\right)\right]$$

$$= \mathrm{Re}\left[ATe^{i(kx-\omega t)}\right]$$

Transmitted wave moving in $+x$ direction, amplitude AT.

b) At $x=0$, the two strings are attatched.

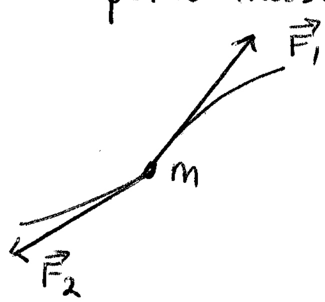
$$\Rightarrow y_0(x=0,t) = y_1(x=0,t)$$

At $x=0$, we have a point mass of mass m.

Newtons second law $\Rightarrow$ if F is the force on m, in the y direction,

$$F = m\left.\frac{d^2y_0}{dt^2}\right|_{x=0} = \left.\frac{md^2y_1}{dt^2}\right|_{x=0}$$

But F is the force arising from the two strings acting on the point mass.



By the small angle approximation, the horizontal components of $\vec{F}_1$, $\vec{F}_2$ are both equal to τ in magnitude.

So the vertical components are

$$F_{1v} = \tau \frac{\partial y_1}{\partial x}\bigg|_{x=0}$$

$$F_{2v} = \tau \left(- \frac{\partial y_0}{\partial x}\bigg|_{x=0} \right)$$

Hence $$\tau \left\{ \frac{\partial y_1(x,t)}{\partial x}\bigg|_{x=0} - \frac{\partial y_0(x,t)}{\partial x}\bigg|_{x=0} \right\} = m \frac{\partial^2 y_1(x,t)}{\partial t^2}\bigg|_{x=0}.$$

c) $$y_0(x=0,t) = \mathrm{Re}\left[Ae^{-i\omega t}(1+R)\right]$$
$$y_1(x=0,t) = \mathrm{Re}\left[ATe^{-i\omega t}\right]$$

$$\Rightarrow \quad \mathrm{Re}\left[ATe^{-i\omega t}\right] = \mathrm{Re}\left[Ae^{-i\omega t}(1+R)\right]$$
$$0 = \mathrm{Re}\left[Ae^{-i\omega t}(1+R-T)\right]$$

For this to be true for all time,

$$\underline{1+R = T} \qquad (*)$$

$$\frac{\partial y_0}{\partial x}\bigg|_{x=0} = \mathrm{Re}\left[Ae^{-i\omega t}(ik - ikR)\right]$$

$$\frac{\partial y_1}{\partial x}\bigg|_{x=0} = \mathrm{Re}\left[AT(ik)e^{-i\omega t}\right]$$

$$\frac{\partial^2 y_1}{\partial t^2}\bigg|_{x=0} = \mathrm{Re}\left[AT(-\omega^2)e^{-i\omega t}\right]$$

$$\Rightarrow \quad \tau\,\mathrm{Re}\left[ikAe^{-i\omega t}(T-(1-R))\right] = m\,\mathrm{Re}\left[-\omega^2 ATe^{-i\omega t}\right]$$

$$Re\left[Ae^{-i\omega t}\left\{ik\tau(T+R-1)+\omega^2 mT\right\}\right]=0$$

For this to be true for all time,

$$\underline{ik\tau(T+R-1)+\omega^2 mT \quad = 0}$$

Substituting for R from (*),

$$ik\tau(T-1+(T-1))+\omega^2 mT=0$$

$$T\{2ik\tau+\omega^2 m\} \quad = 2ik\tau$$

$$T = \frac{2ik\tau}{2ik\tau+m\omega^2}$$

$$T = \frac{+4k^2\tau^2+2ik\tau m\omega^2}{4k^2\tau^2+m^2\omega^4}$$

$$\Rightarrow \quad R = \frac{-m^2\omega^4+2im\tau\omega^2 k}{4k^2\tau^2+m^2\omega^4}$$

The real amplitude of the incident wave is $|A|$, of the transmitted wave is $|AT|$ and of the reflected wave $|AR|$.

So the amplitude of the transmitted wave over that of the incident wave is $\frac{|AT|}{|A|} = \frac{|A||T|}{|A|}$

$$= |T|$$

The amplitude of the reflected wave over the amplitude of the incident wave is $\frac{|AR|}{|A|} = |R|$

The energy of a wave is proportional to the square of its real amplitude.

$\Rightarrow$ Energy is conserved if $1 = |R|^2+|T|^2$.

$$|R|^2 = \frac{m^4\omega^8 + 4m^2\omega^4k^2\tau^2}{(4k^2\tau^2 + m^2\omega^4)^2}$$

$$|T|^2 = \frac{16k^4\tau^4 + 4k^2\tau^2m^2\omega^4}{(4k^2\tau^2 + m^2\omega^4)^2}$$

$$|R|^2 + |T|^2 = \frac{16k^4\tau^4 + 8k^2\tau^2m^2\omega^4 + m^4\omega^8}{(4k^2\tau^2 + m^2\omega^4)^2}$$

$$\underline{|R|^2 + |T|^2 = 1}$$ So energy is conserved. ☒

2. $y_0(x<a,t) = \mathrm{Re}\{e^{-i\omega t}(Be^{-ik_0x} + Ce^{ik_0x})\}$

$y_1(x>a,t) = \mathrm{Re}\{Ae^{-i\omega t}(e^{-ik_1(x-a)} + re^{ik_1(x-a)})\}$

2.1. a) The $x=0$ boundary is a fixed boundary; i.e. $y_0(x=0,t) = 0 \quad \forall t$.

$$\Rightarrow \quad 0 = \mathrm{Re}\{e^{-i\omega t}(Be^{-ik_0x} + Ce^{ik_0x})\}\big|_{x=0}$$

For this to be true for all time,

$$0 = (Be^{-ik_0x} + Ce^{ik_0x})\big|_{x=0}$$

$$0 = B + C$$

$$B = -C.$$

$$\Rightarrow \quad y_0(x<a,t) = \mathrm{Re}\{e^{-i\omega t}B(e^{-ik_0x} - e^{ik_0x})\}$$

$$= \mathrm{Re}\{e^{-i\omega t}B(-2i\sin(k_0x))\}$$

Let $\underline{D = -2iB = 2iC}$.

Then $\underline{y_0(x<a,t) = \mathrm{Re}\{e^{-i\omega t}D\sin(k_0x)\}}$

b) There are two conditions that must be satisfied at $x=a$.

The first is that the two strings are connected at $x=a$.

This is equivalent to the condition

$$\underline{\lim_{x\to a} y_0(x<a,t) = \lim_{x\to a} y_1(x>a,t)}$$

The second condition is that the gradient of the two strings is continuous at $x=a$. Otherwise, there would be a non-zero force acting on an arbitrarily small element of string, resulting in arbitrarily large accelerations.

So $$\underline{\lim_{x\to a} \frac{\partial y(x<a,t)}{\partial x} = \lim_{x\to a} \frac{\partial y(x>a,t)}{\partial x}}$$

c) The functions we are dealing with are analytic, so the limits are obvious;

$$\lim_{x\to a} y_0(x<a,t) = \mathrm{Re}\{e^{-i\omega t}D\sin(k_0a)\}$$

$$\lim_{x\to a} y_1(x>a,t) = \mathrm{Re}\{e^{-i\omega t}A(1+r)\}$$

$$\frac{\partial y_0}{\partial x}(x<a,t) = Re\left\{ e^{-i\omega t} D k_0 \cos(k_0 x) \right\}$$

$$\Rightarrow \quad \lim_{x\to a} \frac{\partial y_0}{\partial x} = Re\left\{ e^{-i\omega t} D k_0 \cos(k_0 a) \right\}$$

$$\frac{\partial y_1}{\partial x}(x>a,t) = Re\left\{ A e^{-i\omega t}\left(-ik_1 e^{-ik_1(x-a)} + ik_1 r e^{ik_1(x-a)}\right) \right\}$$

$$\Rightarrow \quad \lim_{x\to a} \frac{\partial y_1}{\partial x} = Re\left\{ A e^{-i\omega t}(ik_1)(r-1) \right\}$$

So our two conditions from part b) become

$$Re\left\{ e^{-i\omega t} D \sin(k_0 a) \right\} = Re\left\{ e^{-i\omega t} A(1+r) \right\} \qquad (1)$$

and $$Re\left\{ e^{-i\omega t} D k_0 \cos(k_0 a) \right\} = Re\left\{ ik_1 A e^{-i\omega t}(r-1) \right\} \qquad (2)$$

Rearranging (1), we get

$$Re\left\{ e^{-i\omega t}\left(D\sin(k_0 a) - A(1+r)\right) \right\} = 0$$

For this to be true for all time, we have

$$D\sin(k_0 a) = A(1+r) \qquad (3)$$

Rearranging (2), we get

$$Re\left\{ e^{-i\omega t}\left(D k_0 \cos(k_0 a) - ik_1 A(1-r)\right) \right\} = 0$$

Again, for this to be true for all time,

$$D k_0 \cos(k_0 a) = ik_1 A(r-1) \qquad (4)$$

Substituting for A from (3) into (4),

$$D k_0 \cos(k_0 a) = ik_1 (r-1) \cdot \frac{D\sin(k_0 a)}{1+r}$$

$$(1+r) k_0 \cos(k_0 a) = ik_1 (r-1) \sin(k_0 a)$$

$$r\left[k_0 \cos(k_0 a) - ik_1 \sin(k_0 a)\right] = -ik_1 \sin(k_0 a) - k_0 \cos(k_0 a)$$

$$r = -\frac{\left(k_0 \cos(k_0 a) + ik_1 \sin(k_0 a)\right)^2}{k_0^2 \cos^2(k_0 a) + k_1^2 \sin^2(k_0 a)}$$

$$r = \frac{k_1^2 \sin^2(k_0 a) - k_0^2 \cos^2(k_0 a) - ik_0 k_1 \sin(2k_0 a)}{k_1^2 \sin^2(k_0 a) + k_0^2 \cos^2(k_0 a)}$$

Finally, to convert this to the desired form, note that ω is the same for both strings. $\Rightarrow$ $c_0 k_0 = c_1 k_1$

$$k_0 = \frac{c_1 k_1}{c_0}$$

$$\Rightarrow \quad r = \frac{k_1^2 \sin^2\left(\frac{c_1 k_1 a}{c_0}\right) - \left(\frac{c_1^2 k_1^2}{c_0^2}\right)\cos^2\left(\frac{c_1 k_1 a}{c_0}\right) - i\frac{c_1 k_1^2}{c_0}\sin\left(\frac{2c_1 k_1 a}{c_0}\right)}{\left(\frac{c_1 k_1}{c_0}\right)^2 \cos^2\left(\frac{c_1 k_1 a}{c_0}\right) + k_1^2 \sin^2\left(\frac{c_1 k_1 a}{c_0}\right)}$$

$$r = \frac{c_0^2 \sin^2\left(\frac{c_1 k_1 a}{c_0}\right) - c_1^2 \cos^2\left(\frac{c_1 k_1 a}{c_0}\right) - i c_0 c_1 \sin\left(\frac{2c_1 k_1 a}{c_0}\right)}{c_1^2 \cos^2\left(\frac{c_1 k_1 a}{c_0}\right) + c_0^2 \sin^2\left(\frac{c_1 k_1 a}{c_0}\right)}$$

2.2. $$R_{01} = \frac{Z_0 - Z_1}{Z_0 + Z_1}$$

$$T_{01} = \frac{2Z_0}{Z_0 + Z_1}$$

a) The above expressions are for going from the light string 0 to the heavy string 1. By symmetry, the expressions for going the other way simply involve switching $Z_0 \leftrightarrow Z_1$.

$$R_{10} = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \qquad T_{10} = \frac{2Z_1}{Z_1 + Z_0}$$

b) By the definition of the reflection coefficient, if the incoming wave has ~~any~~ (complex) amplitude A, then the wave that is immediately reflected at $x=a$ has amplitude $\underline{R_{10}A}$

The wave that enters the light string, reflects at $x=0$ and then enters the heavy string again has amplitude $\underline{-T_{10}T_{01}e^{2ik_0a}A}$

(The two factors of T_{10} and T_{01} correspond to the two transmissions through $x=a$; the $-$ sign arises from the reflection from the fixed $x=0$ point; and the e^{2ik_0a} arises from the path difference between the first wave and the second.)

c) Generalising, a wave that enters the light string, is reflected at $x=0$ n times and then reenters the heavy string will have (complex) amplitude $\quad T_{10} T_{01} (-e^{2ik_0 a})^n (R_{01})^{n-1} A$

So the complex amplitude of the reflected wave in total is

$$A_{ref} = R_{10} A + \sum_{n=1}^{\infty} T_{10} T_{01} (-e^{2ik_0 a})^n (R_{01})^{n-1} A$$

Using the expression in the question to evaluate the sum,

$$A_{ref} = R_{10} A + \frac{T_{10} T_{01} (-e^{2ik_0 a}) A}{1 - (-e^{2ik_0 a}) R_{01}}$$

$$A_{ref} = \frac{R_{10}(1 + R_{01} e^{2ik_0 a}) - T_{10} T_{01} e^{2ik_0 a}}{1 + R_{01} e^{2ik_0 a}} A$$

So the reflection coefficient is

$$r = \frac{\left(R_{10}(1 + R_{01} e^{2ik_0 a}) - T_{10} T_{01} e^{2ik_0 a}\right)(1 + R_{01} e^{-2ik_0 a})}{1 + (R_{01})^2 + 2R_{01}\cos(2k_0 a)}$$

$$r = \frac{R_{10}\left(1 + (R_{01})^2 + 2R_{01}\cos(2k_0 a)\right) - T_{10} T_{01} e^{2ik_0 a} - T_{10} T_{01} R_{01}}{1 + (R_{01})^2 + 2R_{01}\cos(2k_0 a)}$$

$$r = \frac{(Z_1 - Z_0)\left[(Z_0+Z_1)^2 + (Z_0 - Z_1)^2 + 2(Z_0+Z_1)(Z_0-Z_1)\cos(2k_0 a)\right] - 4Z_0 Z_1\left\{(Z_0+Z_1)e^{2ik_0 a} - (Z_0 - Z_1)\right\}}{(Z_0+Z_1)^3 + (Z_0-Z_1)^2(Z_0+Z_1) + 2(Z_0+Z_1)^2(Z_0-Z_1)\cos(2k_0 a)}$$

$$= \frac{2(Z_1 - Z_0)\left[Z_0^2 + Z_1^2 + (Z_0^2 - Z_1^2)\cos(2k_0 a)\right] - 4Z_0 Z_1\left[(Z_0+Z_1)(\cos(2k_0 a) + i\sin(2k_0 a)) - (Z_0 - Z_1)\right]}{2(Z_0+Z_1)\left(Z_0^2 + Z_1^2 + (Z_0^2 - Z_1^2)\cos(2k_0 a)\right)}$$

$$= \frac{(Z_1 - Z_0)\left[Z_0^2\cos^2(k_0 a) + Z_1^2\sin^2(k_0 a)\right] - Z_0 Z_1\left[2Z_1\cos^2(k_0 a) - 2Z_0\sin^2(k_0 a) + i(Z_0+Z_1)\sin(2k_0 a)\right]}{(Z_0+Z_1)\left(Z_0^2\cos^2(k_0 a) + Z_1^2\sin^2(k_0 a)\right)}$$

$$r = \frac{Z_1^2\sin^2(k_0 a) - Z_0^2\cos^2(k_0 a) - iZ_0 Z_1\sin(2k_0 a)}{Z_0^2\cos^2(k_0 a) + Z_1^2\sin^2(k_0 a)}$$

$Z = \mu c = \frac{T}{c}$, where T is the tension in the string. But as the tension must be equal in each string, it is simple to see that the above is equivalent to the result from 3.1.

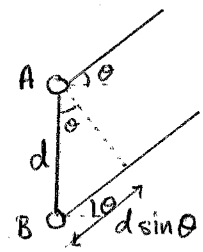
3. a) Intensity from Tower A is I_A, Tower B is I_B.

Hence the corresponding amplitudes $A_A \propto \sqrt{I_A}$, $A_B \propto \sqrt{I_B}$.

Choose $t=0$ so that the signal received from tower A is

$$Re\{A_A e^{i\omega t}\} \propto Re\{\sqrt{I_A} e^{i\omega t}\}.$$

The signals from both towers are in phase. So the only phase difference between the received signals arises from the waves travelling different distances to the ~~receiver~~ receiver.



As the receiver is a "long" distance from the towers, we can approximate the waves from the towers as parallel. Hence the path difference is $d\sin\theta$ and the phase difference $\frac{2\pi d\sin\theta}{\lambda} = kd\sin\theta \equiv \Delta\phi(\theta)$.

So the recieved signal from tower B is

$$Re\{A_B e^{i(\omega t + \Delta\phi(\theta))}\} \propto Re\{\sqrt{I_B} e^{i(\omega t + \Delta\phi(\theta))}\}$$

And the total signal is

$$Re\{[A_A + A_B e^{i\Delta\phi(\theta)}]e^{i\omega t}\} \propto Re\{[\sqrt{I_A} + \sqrt{I_B} e^{i\Delta\phi(\theta)}]e^{i\omega t}\}$$

Hence, the recieved intensity is

$$I(\theta) = |[\sqrt{I_A} + \sqrt{I_B} e^{i\Delta\phi(\theta)}]e^{i\omega t}|^2$$

$$= (\sqrt{I_A} + \sqrt{I_B} e^{i\Delta\phi(\theta)})(\sqrt{I_A} + \sqrt{I_B} e^{-i\Delta\phi(\theta)})|e^{i\omega t}|^2$$

$$= I_A + I_B + \sqrt{I_A I_B}(e^{i\Delta\phi(\theta)} + e^{-i\Delta\phi(\theta)})$$

$$= I_A + I_B + 2\sqrt{I_A I_B}\cos(\Delta\phi(\theta))$$

$$= (\sqrt{I_A} - \sqrt{I_B})^2 + 4\sqrt{I_A I_B}\cos^2\left(\frac{\Delta\phi(\theta)}{2}\right)$$

$$\underline{I(\theta) = (\sqrt{I_A} - \sqrt{I_B})^2 + 4\sqrt{I_A I_B}\cos^2\left[\tfrac{1}{2}kd\sin\theta\right]}$$

b) For real x, $0 \le \cos^2 x \le 1$.

So $I_{max}(\theta)$ occurs when the $\cos^2$ function equals 1;

$$\underline{I_{max} = (\sqrt{I_A} + \sqrt{I_B})^2 = I_A + I_B + 2\sqrt{I_A I_B}}$$

The values of x for which $\cos^2 x = 1$ are $x = n\pi,\ n \in \mathbb{Z}$.

So the values of θ for which I is a maximum are

$$\frac{1}{2} kd\sin\theta = n\pi$$

$$\underline{\sin\theta = \frac{n\lambda}{d}} \quad \text{or} \quad \underline{\theta = \arcsin\left(\frac{n\lambda}{d}\right)}$$

c) As $0 \leq \cos^2 x \leq 1$, I_{min} occurs when the $\cos^2$ function is zero.

$$\underline{I_{min} = (\sqrt{I_A} - \sqrt{I_B})^2 = I_A + I_B - 2\sqrt{I_A I_B}}$$

$\cos^2 x = 0 \Rightarrow x = (m+\frac{1}{2})\pi,\ m \in \mathbb{Z}$.

So the values of θ for which I is a minimum are

$$\frac{1}{2} k d \sin\theta = (m+\frac{1}{2})\pi$$

$$\underline{\sin\theta = (m+\frac{1}{2})\frac{\lambda}{d}} \quad \text{or} \quad \underline{\theta = \arcsin\left[(m+\frac{1}{2})\frac{\lambda}{d}\right]}$$

d) If the two towers are not in phase, all that changes in the above analysis ~~is~~ is the function $\Delta\phi(\theta)$.

(Specifically, for perfectly out of phase waves,

$$\Delta\phi(\theta) \to \Delta\phi'(\theta) \equiv \Delta\phi(\theta) + \pi$$

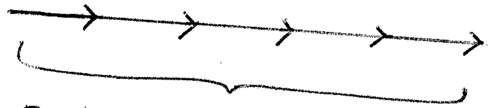
In general, $\Delta\phi(\theta) \to \Delta\phi''(\theta) \equiv \Delta\phi(\theta) + \alpha, \quad 0 \leq \alpha < 2\pi$).

Hence, the expression for the intensity will have essentially the same form;

$$I(\theta) = (\sqrt{I_A} - \sqrt{I_B})^2 + 4\sqrt{I_A I_B}\cos^2(\Delta\phi''(\theta))$$

Thus we immediately see that the maximum and minimum values of I will <u>not</u> change, but the values of θ at which the occur <u>will</u>.

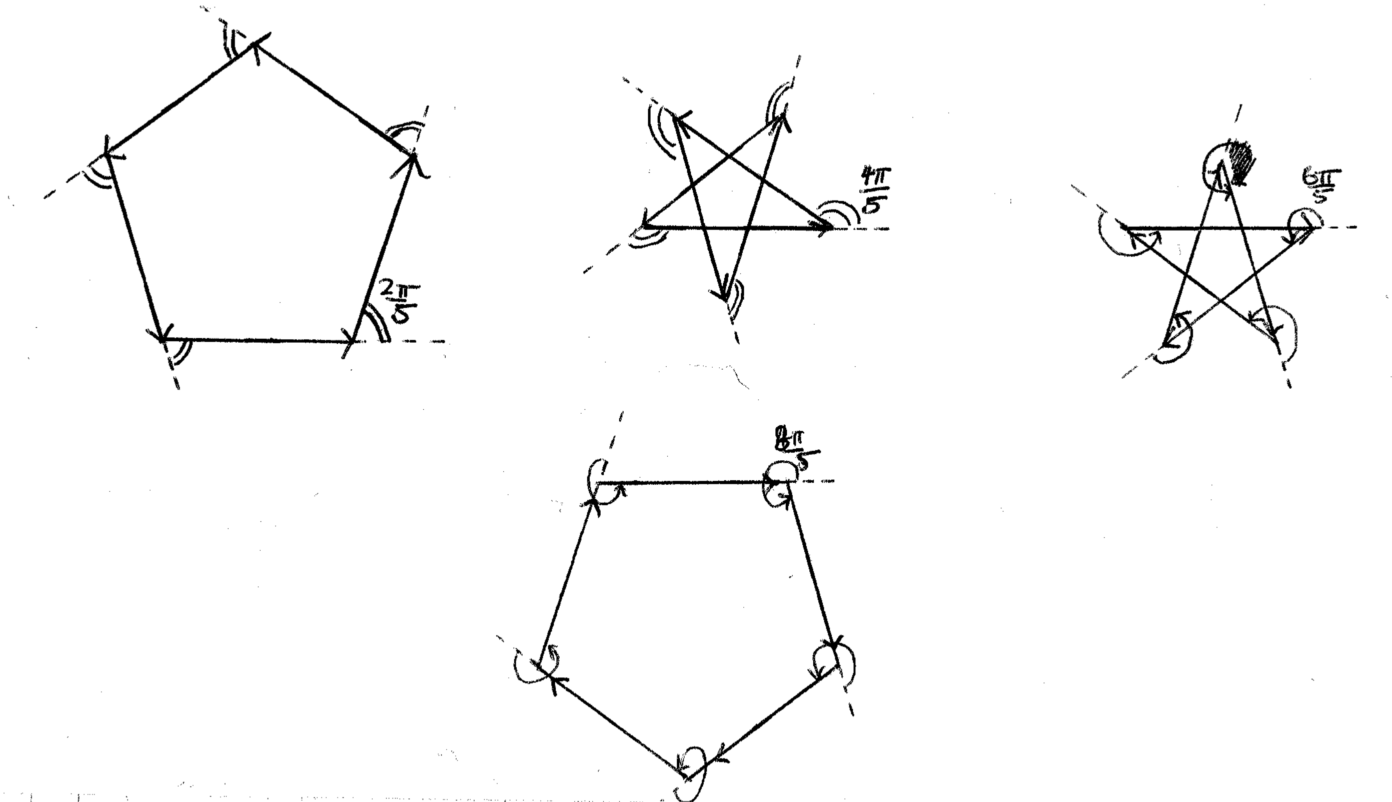
4. a) The principal maximum occurs when waves from each slit arrive in phase.



Each element equal in length, and all colinear.

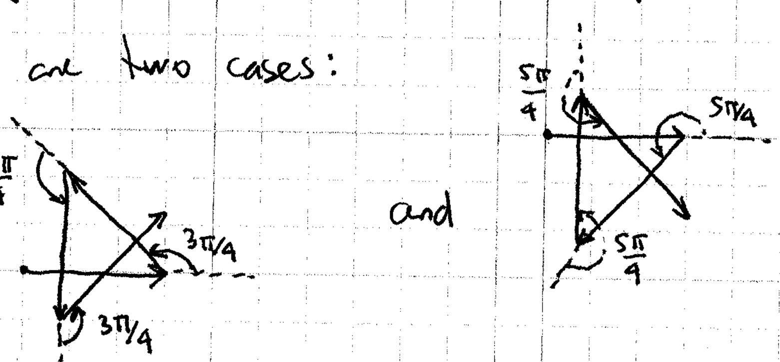
b) The minima for 5 equal slits occur when the phase difference between adjacent slits is $\frac{2\pi}{5}n$, $n \in [1,2,3,4]$

So there are 4 cases. In each diagram, each individual segment has equal magnitude.

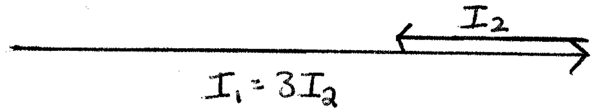


c) The secondary maxima for 4 slits occur when the phase difference between adjacent slits is $3\pi/4$ or $5\pi/4$ (see LN "Interference & Diffraction", p. 13). There are two cases:

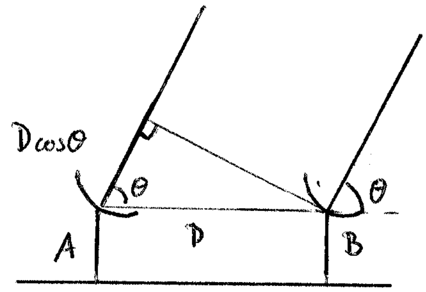
(Each segment has the same magnitude.)



d) The minimum will occur when the phase difference between the adjacent slits is π.



5. a) As the star is far away, light received by the two detectors from the star can be considered essentially parallel.



From the diagram, we see that the path difference is $\underline{D\cos\Theta}$.

b) The signal received from each detector comes from the same source, so the only phase difference is due to the path difference.
Each detector individually would register an intensity of I_0.

So if the signal at B is $Re\{A_1 e^{i\omega t}\} \propto Re\{\sqrt{I_0}\, e^{i\omega t}\}$

Then the signal at A is $Re\{A e^{i(\omega t + \Delta\phi(\theta))}\} \propto Re\{\sqrt{I_0}\, e^{i(\omega t + \Delta\phi(\theta))}\}$

(where $\Delta\phi(\theta) = \frac{2\pi D\cos\Theta}{\lambda}$).

The combined signal is $Re\{ A e^{i\omega t}(1 + e^{i\Delta\phi(\theta)})\} \propto Re\{\sqrt{I_0}\, e^{i\omega t}(1+e^{i\Delta\phi(\theta)})\}$

So the intensity is
$$I(\theta) = |\sqrt{I_0}\, e^{i\omega t}(1+e^{i\Delta\phi(\theta)})|^2$$
$$= I_0(2 + 2\cos(\Delta\phi(\theta)))$$
$$= 4I_0\cos^2(\tfrac{1}{2}\Delta\phi(\theta))$$
$$\underline{I(\theta) = 4I_0\cos^2\left(\frac{\pi D}{\lambda}\cos\theta\right)}$$

c) $0 \le \cos^2 x \le 1$ if x is real.

So I is maximal when $\cos\left(\frac{\pi D}{\lambda}\cos\theta\right) = \pm 1$,
$$\frac{\pi D}{\lambda}\cos\theta = \pi n \qquad \text{for } n \in \mathbb{Z}$$
$$\underline{\cos\theta = \frac{n\lambda}{D}}$$

Likewise, I is minimal when $\cos\left(\frac{\pi D}{\lambda}\cos\theta\right) = 0$
$$\frac{\pi D}{\lambda}\cos\Theta = (m+\tfrac{1}{2})\pi, \quad m \in \mathbb{Z}$$
$$\underline{\cos\Theta = (m+\tfrac{1}{2})\frac{\lambda}{D}}$$

d) For ~~to~~ the intensity to change from a maximum to a minimum, $\cos\theta$ need change only by $\underline{\frac{\lambda}{2D}}$.

e) $D \approx 1\,m$, $\lambda \approx 10^{-6}\,m$.

So $\frac{\lambda}{2D} \approx 5\times10^{-7}$; that is, it is easy to notice a shift in the stars position if $\cos\theta$ changes by 5×10^{-7}.

$\cos\theta$ changes the least with θ when $\cos\theta = \pm 1$.

If $\cos\theta = 1$ and $\cos(\theta + \delta\theta) = 1 - 5\times10^{-7}$,

then $\underline{\delta\theta = 10^{-3}\,rad = 0{\cdot}06^\circ}$

Thus the sensitivity offered is to motions of at worst the above scale. This is equivalent to a shift of 1mm at a distance of 1m, i.e. effectively unnoticeable by the unaided human eye.