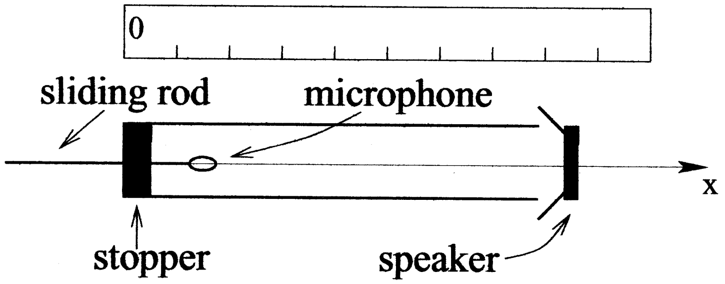


Figure 1: Modification of sound tube experiment from Lab I.

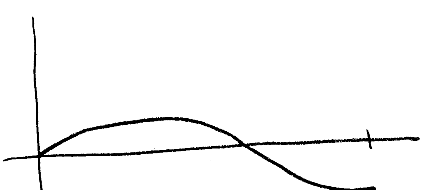
1 Lab Experiment I [25 points]

This problem considers a modification of the sound tube experiment from Lab I. You now *close* one end of the tube with a stopper and mount the microphone on a rod which slides through the stopper, thus allowing measurements of *pressure* variations at various positions. Finally, you measure position from the left side of the stopper. (See figure above.)
Note: The new tube has a different length from those used in lab.

(a) Resonant frequencies (6 points)

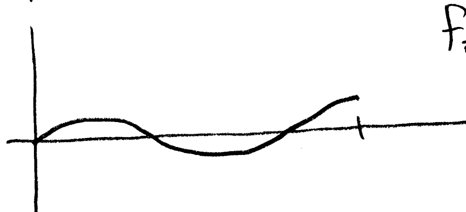
Tuning the sound generator, you find that certain frequencies result in an audible sound from the tube and a maximum in the microphone signal near the stopper end of the tube. Suppose that the *lowest* frequency at which you observe this phenomenon is at $f_1 \approx 115$ Hz. Determine the *two* lowest frequencies (f_2 and f_3) above f_1 at which you would expect to observe the same phenomena.
Note: There is additional space on the next page for your work.

f_1: $\frac{1}{4}\lambda = L'$, $\lambda = 4L'$, $f_1 = \frac{c}{\lambda} = \frac{c}{4L'}$



f_2: $\frac{3}{4}\lambda = L'$, $\lambda = \frac{4L}{3}$, $f_2 = \frac{c}{\lambda} = \frac{c}{4L'} \cdot 3$

$\Rightarrow f_2 = 3f_1 = \boxed{345\text{ Hz} = f_2}$



f_3: $\frac{5}{4}\lambda = L'$, $\lambda = \frac{4L}{5}$, $f_3 = \frac{c}{\lambda} = \frac{c}{4L} \cdot 5$

$\Rightarrow f_3 = 5f_1 = \boxed{575\text{ Hz} = f_3}$

(b) Third mode (8 points)

On the axes below, *sketch* the sound displacement and pressure patterns you would expect when the sound generator is set to f_3 (the "third mode"). On *both* sketches, *indicate* on *both* boundaries either the value or slope of your curve (whichever is determined by the boundary conditions).

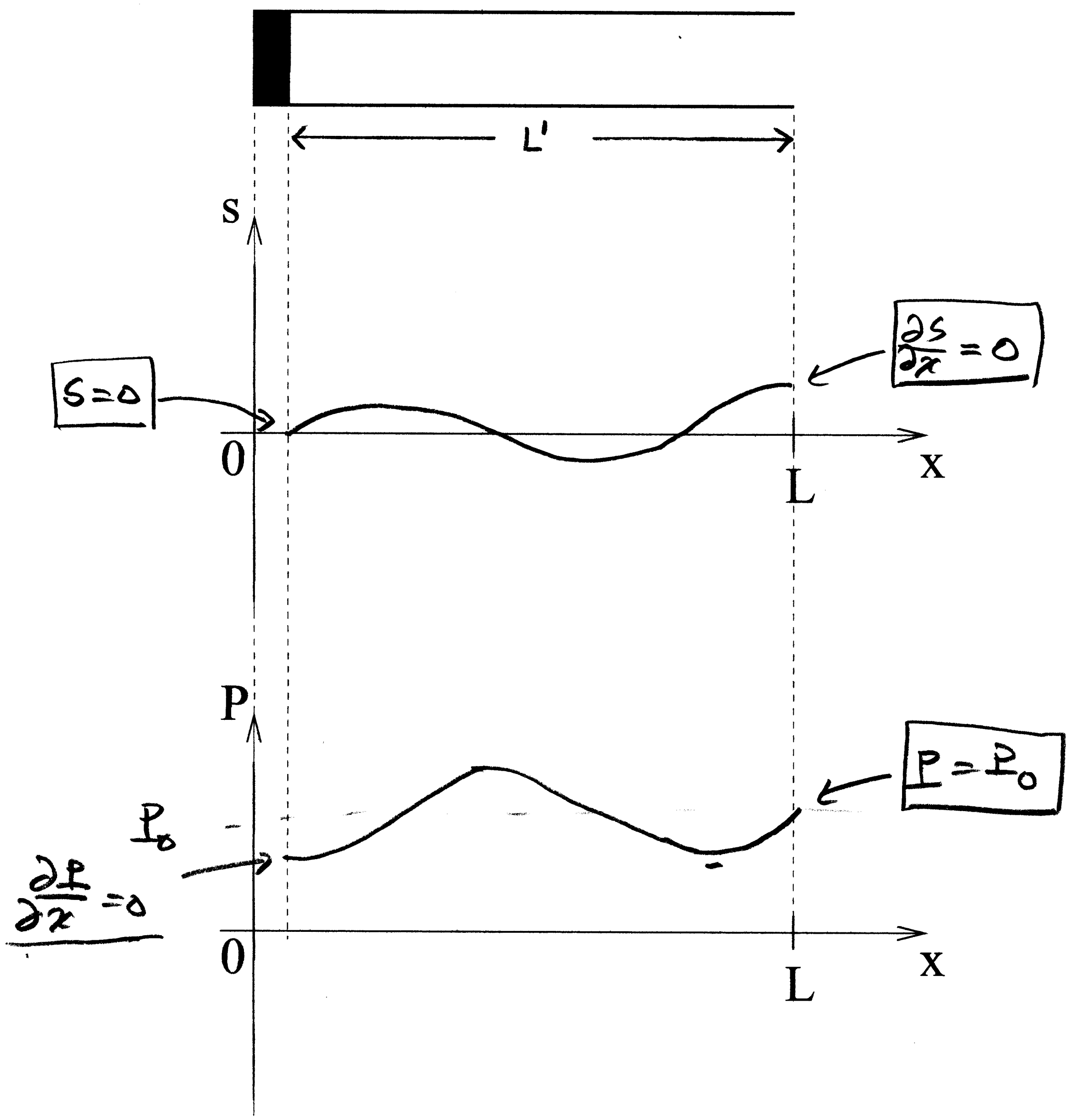


Figure 2: Sketches of sound displacement and pressure for third normal mode

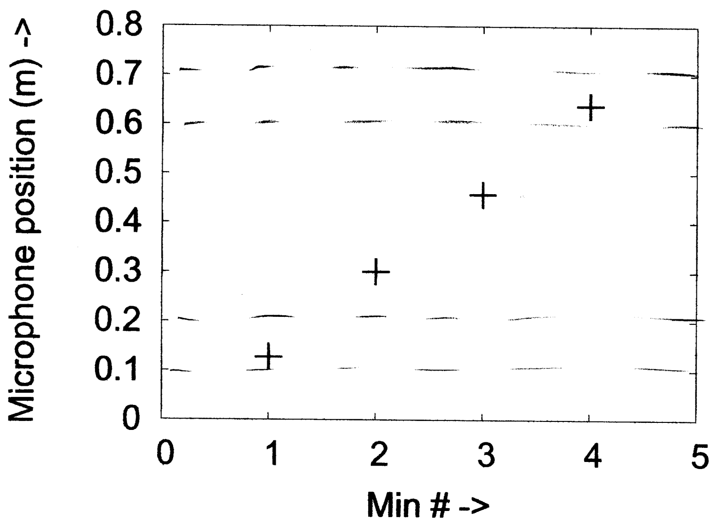


Figure 3: Locations of minimum microphone signal for the fifth resonance.

(c) Speed of sound (7 points)

The figure above shows measured positions of minimum microphone signal for the fifth resonance, $f_5 \approx 1020$ Hz. From these data, *calculate the speed of sound in units of m/s.*

$\frac{\lambda}{2}$ = dist between mins $\approx \frac{0.65 - 0.15}{3}$ m $\approx \frac{0.5\text{ m}}{3}$

$\Rightarrow \lambda = \frac{1\text{ m}}{3}$

$c = f \cdot \lambda = 1020\text{ Hz} \cdot \frac{1\text{ m}}{3} = \boxed{340\text{ m/s} = c}$

(d) Challenge: width of the stopper (4 points)

Given that the data in Figure (c) were taken with the zero of the ruler aligned with the left end of the stopper, *estimate* the length of the stopper in units of cm. *Give your reasoning.*

1st min is $\frac{1}{4}\lambda \approx \frac{0.33\text{m}}{4} \approx 0.08$ m from right end.

1st min is at 0.12 m from left end.

Thus, $\boxed{\text{stopper width} \approx 0.12\text{ m} - 0.08\text{ m} = 0.04\text{ m} = 4\text{ cm}}$

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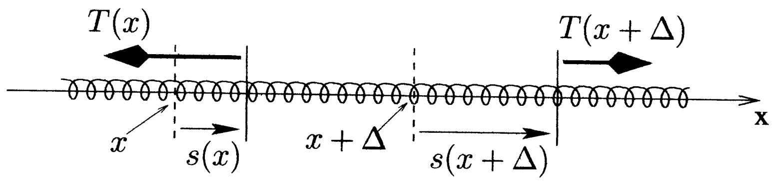

Figure 4: Analysis of longitudinal motion along a spring.

2 Waves on a spring [25 points]

Figure 2 shows a section of a spring which is undergoing longitudinal motion. The entire spring is stretched between two walls and has total mass M, length L and spring constant K. As with sound, the appropriate degrees of freedom are $s(x,t)$, defined as giving the *displacement* along x of the chunk which originated at x. Finally, the function $T(x)$ in the figure represents the tension in the spring acting on the chunk which originated point x.

(a) Physics informed guess

The wave speed c of this system is proportional to the spring constant to some power, $c \propto K^\alpha$. Based on the discussion in lecture, make an informed guess for the exponent α. Explain your answer briefly (one or two sentences).

Usually, $c = \sqrt{\frac{\text{stiffness}}{\text{inertia}}}$. K measures stiffness, thus $c \propto \sqrt{K}$, $\boxed{\alpha = \frac{1}{2}}$

Also, $c \propto M^\beta$. Make an informed guess for the exponent β. *Explain briefly.*

Now, M measures inertia. Thus $c \propto \sqrt{\frac{1}{M}}$, $\boxed{\beta = -\frac{1}{2}}$

Finally, the wave speed depends on L. Using your values for α and β above, determine the exponent γ which ensures that the *guess* $c = K^\alpha M^\beta L^\gamma$ has the right units.

Units of $c = \frac{m}{s}$ = Units of $K^{1/2} M^{-1/2} L^\gamma = \sqrt{\frac{K}{M}} L^\gamma = Hz \cdot m^\gamma = m^\gamma/s$

Thus, $\boxed{\gamma = 1}$

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(b) Law of motion, differential form

Determine $\frac{\partial^2 s(x,t)}{\partial t^2}$ in terms of no quantities other than K, M, L, and $T(x)$ or any of its derivatives.
Hint: Consider the chunk of spring in the figure which originated between points x and $x+\Delta$, and then take the limit $\Delta \to 0$.

$$\sum F_x = m_{ch}\, a_{cm,x} \qquad \text{mass/length} = M/L$$

$$\frac{T(x+\Delta) - T(x)}{\Delta} = \mu \cdot \cancel{\Delta} \cdot a_{cm,x}$$

$$\lim_{\Delta \to 0} : \quad \frac{\partial T}{\partial x} = \frac{M}{L}\frac{\partial^2 s(x,t)}{\partial t^2} \Rightarrow \boxed{\frac{\partial^2 s}{\partial t^2} = \frac{L}{M}\frac{\partial T}{\partial x}}$$

(c) Constitutive relation (7 points)

Each small chunk of length ℓ taken from a spring of total length L and spring constant K will have a tension $T_{ch} = T_0 + K\,\frac{L}{\ell}\,\Delta\ell$, where T_0 is the tension throughout the spring in the absence of waves and $\Delta\ell$ is the *change* in the length of the chunk due to the presence of waves. From this, *determine* $T(x)$, the tension at point x, in terms of no quantities other than T_0, K, M, L, and $s(x,t)$ or any of its derivatives.
Hint: Again consider a tiny chunk between points x and $x+\Delta$.

$$\ell = \Delta$$

$$\Delta\ell = [(\cancel{x}+\cancel{\Delta}+s(x+\Delta)) - (\cancel{x}+s(x))] - \cancel{\Delta}$$

$$\Rightarrow T_{ch} = T_0 + K\,\frac{L}{\Delta}\,(s(x+\Delta) - s(x))$$

$$\lim_{\Delta \to 0} : \boxed{T(x) = T_0 + KL\frac{\partial s}{\partial x}}$$

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Challenge:
(d) Equation of motion

Derive the equation of motion for the interior chunks of the spring.

(c) into (b):

$$\frac{\partial^2 s}{\partial t^2} = \frac{L}{M}\frac{\partial}{\partial x}\left(\cancel{T_0} + KL\frac{\partial s}{\partial x}\right) = \boxed{\frac{KL^2}{M}\frac{\partial^2 s}{\partial x^2} = \frac{\partial^2 s}{\partial t^2}}$$

Give the wave-speed constant c in terms of only M, L and K.
Hint: This should compare favorably with your guess in part (a).

$$\frac{\partial^2 s}{\partial t^2} \equiv c^2\frac{\partial^2 s}{\partial x^2} \Rightarrow c = \sqrt{\frac{KL^2}{M}} = \sqrt{\frac{K}{M}}\,L$$

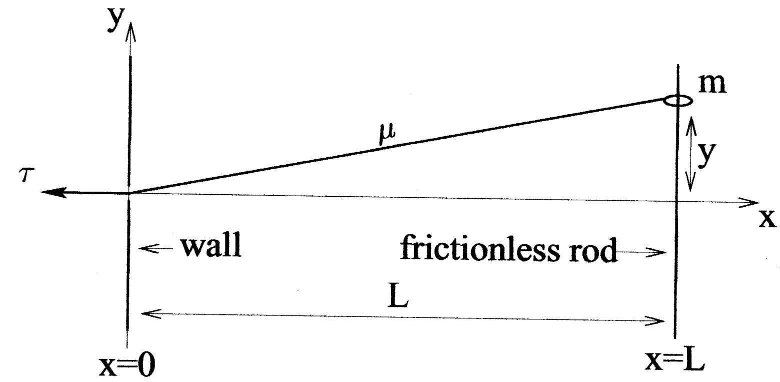

Figure 5: Standard string system with massive boundary condition

3 String with novel boundary condition [25 points]

Figure 3 shows a standard string system from lecture of length L, mass per unit length μ, and applied tension τ. The only modification is that one end is attached to a mass m which is much heavier than the string ($m \gg \mu L$) and slides freely along a frictionless rod. You may ignore the effects of gravity.

(a) Tension forces (5 points)

Determine the x- and y- components of the force $\vec{F}$ which the string exerts on the mass m when, as in the figure, the string is perfectly straight and the mass is at position y. *Express* your answers in terms of no quantities other than μ, τ, m, L and y.

$$\boxed{\begin{aligned} F_x &= -\tau \\ F_y &= -\tau\,(\text{slope}) = -\tau \frac{y}{L} \end{aligned}}$$

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(b) Effective simple harmonic oscillator (5 points)

When released from the position in the figure, the mass oscillates up and down the rod. Because in this case the string is relatively light ($\mu L \ll m$), it remains approximately straight during this motion. Use your result from **(a)** to *estimate* the angular frequency ω of this motion in terms of no quantities other than μ, τ, m and L.

Hint: Find an effective spring constant for the system.

If it were a spring $F_y = -ky$, thus $k = \frac{\tau}{L}$.

$$\omega = \sqrt{\frac{k}{m}} = \boxed{\sqrt{\frac{\tau}{mL}} = \omega}$$

(c) Boundary conditions (5 points)

Derive the boundary condition which applies to the end of the string at $x = L$, for the general case *even* when the string is *not* straight. Remember that this condition must be in terms of no quantities other than τ, μ, L, m, and the function $y(x,t)$ and its derivatives.

Note: If you are unable to do this part, assume for part **(d)** that the boundary condition is $\frac{\partial^3 y(x=L,t)}{\partial t^3} = -\frac{\tau L}{m}\frac{\partial^3 y(x=L,t)}{\partial t \partial x^2}$, which is *not* the correct boundary condition!

$$\Sigma F_y = m a_y$$

$$-\tau \frac{\partial y}{\partial x}\bigg|_{x=L} = m \frac{\partial^2 y}{\partial t^2}\bigg|_{x=L}$$

$$\boxed{-\tau \frac{\partial y(x=L,t)}{\partial x} = m \frac{\partial^2 y(x=L,t)}{\partial t^2}}$$

(d) Normal modes (5 points)

Derive a mathematical equation in terms of no quantities other than ω, k, m, τ, μ and L which must hold in order that the normal mode solution $y(x,t) = A\sin(kx)\cos(\omega t)$ satisfy the boundary condition from **(c)**.

$$+\tau k \cancel{A} \cos kL \cancel{\cos\omega t} = + m\cancel{A}\sin kL\, \omega^2 \cancel{\cos\omega t}$$

$$\boxed{m \sin kL\, \omega^2 = \tau k \cos kL}$$

(e) Challenge: Lowest mode (5 points)

Attempt this part only if you have your own answer for **(c)**.

When the mass is much heavier than the string, the lowest frequency mode corresponds to a very long wavelength, and thus ~~so that~~ $k \to 0$. Using the facts that $\sin\theta \to \theta$ and $\cos\theta \to 1$ as $\theta \to 0$, *show* that your result in **(d)** predicts the same frequency in this limit as you estimated in **(b)**!

$k \to 0 \Rightarrow$

$$m \cdot \cancel{k} L\, \omega^2 = \tau \cancel{k} \cdot 1$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{\tau}{mL}}} \checkmark$$

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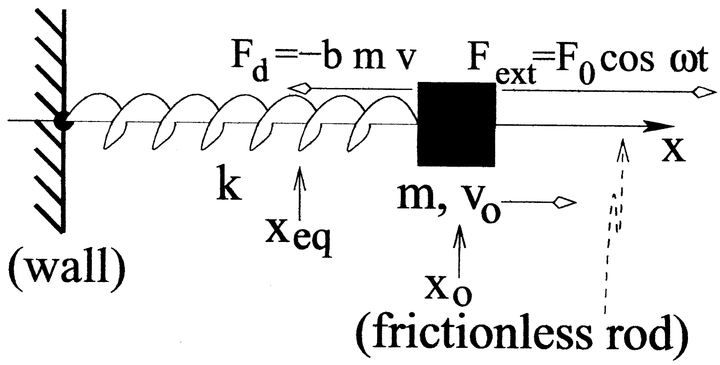

Figure 6: Damped, driven oscillator from lecture

4 Complete theory for driven oscillator [25 points]

Figure 6 shows the damped, driven oscillator studied in lecture. We showed the equation of motion for this system to be

$$-\omega_0^2\,(x - x_{eq}) - b\frac{dx}{dt} + \frac{F_0}{m}\,\mathrm{Re}\,e^{i\omega t} = \frac{d^2x}{dt^2},$$

where $\omega_0 \equiv \sqrt{k/m}$. Also, we showed that $x(t) = x_{eq} + \mathrm{Re}\,(\underline{A}e^{i\omega t})$ satisfies the equation of motion provided

$$\underline{A} = \frac{F_0/m}{\omega_0^2 - \omega^2 + ib\omega}, \tag{4.1}$$

However, because the value of $\underline{A}$ is completely determined, there are no *free* parameters in this solution and it thus cannot be a general solution for the damped, driven oscillator.

(a) General solution (10 points)

When the damping is relatively small but not zero $(0 < b < \omega_0)$, ~~$(b << \omega_0)$~~,

$$x(t) = x_{eq} + \mathrm{Re}\left(\underline{B}e^{\underline{\alpha}t}\right) + \mathrm{Re}\left(\underline{A}e^{i\omega t}\right) \tag{4.2}$$

also satisfies the equation of motion for any value of $\underline{B}$, provided that $\underline{A}$ obeys Eq. (4.1) and $\underline{\alpha}$ satisfies $\underline{\alpha}^2 + b\underline{\alpha} + \omega_0^2 = 0$. Given this information, *explain* briefly (one or two sentences) why Eq. (4.2) is a general solution for the driven oscillator with small damping.

- It gives $x(t)$ explicitly.
- It solves the EqM.
- It has 2 free parameters ($\mathrm{Re}\,\underline{B}$, $\mathrm{Im}\,\underline{B}$) and the EqM is 2nd order in time with one DoF.

(b) Particular solution for given initial conditions (10 points)

Suppose that at time $t = 0$, the initial position and velocity of the mass are x_0 and v_0, respectively. *Find* the real and imaginary parts of $\underline{B}$ in terms of *only* x_0, v_0, x_{eq}, ω, $\mathrm{Re}\,\underline{\alpha}$, $\mathrm{Im}\,\underline{\alpha}$, $\mathrm{Re}\,(\underline{A})$ and $\mathrm{Im}\,(\underline{A})$.

Note: Please don't spend time evaluating $\mathrm{Re}\,\underline{\alpha}$, $\mathrm{Im}\,\underline{\alpha}$, $\mathrm{Re}\,(\underline{A})$ or $\mathrm{Im}\,(\underline{A})$. Simply leave your answer in terms of these quantities.

$$x_0 = x(t=0) = x_{eq} + \mathrm{Re}\,\underline{B}\,\cancel{e^{\underline{\alpha}\cdot 0}}^{\,1} + \mathrm{Re}\,\underline{A}\,\cancel{e^{i\omega\cdot 0}}^{\,1}$$

$$\Rightarrow \boxed{\mathrm{Re}\,\underline{B} = x_0 - x_{eq} - \mathrm{Re}\,\underline{A}}$$

$$v_0 = \frac{dx(t=0)}{dt} = \mathrm{Re}\left(\underline{\alpha}\,\underline{B}\,\cancel{e^{\underline{\alpha}\cdot 0}}^{\,1}\right) + \mathrm{Re}\left(i\omega\underline{A}\,\cancel{e^{i\omega\cdot 0}}^{\,1}\right)$$

$$= \mathrm{Re}\,\underline{\alpha}\cdot\mathrm{Re}\,\underline{B} - \mathrm{Im}\,\underline{\alpha}\cdot\mathrm{Im}\,\underline{B} - \omega\,\mathrm{Im}\,\underline{A}$$

$$\Rightarrow \mathrm{Im}\,\underline{B} = \frac{\mathrm{Re}\,\underline{\alpha}\cdot\mathrm{Re}\,\underline{B} - \omega\,\mathrm{Im}\,\underline{A} - v_0}{\mathrm{Im}\,\underline{\alpha}}$$

now substitute for $\mathrm{Re}\,\underline{B}$:

$$\boxed{\mathrm{Im}\,\underline{B} = \frac{(\mathrm{Re}\,\underline{\alpha})(x_0 - x_{eq} - \mathrm{Re}\,\underline{A}) - \omega\,\mathrm{Im}\,\underline{A} - v_0}{\mathrm{Im}\,\underline{A}}}$$

(c) Challenge: Long term behavior (5 points)

Explain briefly (one or two sentences) why in lecture it was okay to ignore the $\mathrm{Re}\,(\underline{B}e^{\underline{\alpha}t})$ term in Eq. (4.2) when determining the *final* amplitude and phase of the motion.

The $\mathrm{Re}\,\underline{B}e^{\underline{\alpha}t}$ term decays exponentially with time and $\to 0$ for large times. We know this because $\underline{\alpha} = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4\omega_0^2}}{2} = -\frac{b}{2} \pm i\sqrt{\omega_0^2 - (\frac{b}{2})^2}$ and so $e^{\underline{\alpha}t} = \underline{e^{-\frac{b}{2}t}}\, e^{i\sqrt{\omega_0^2 - (\frac{b}{2})^2}\,t}$.