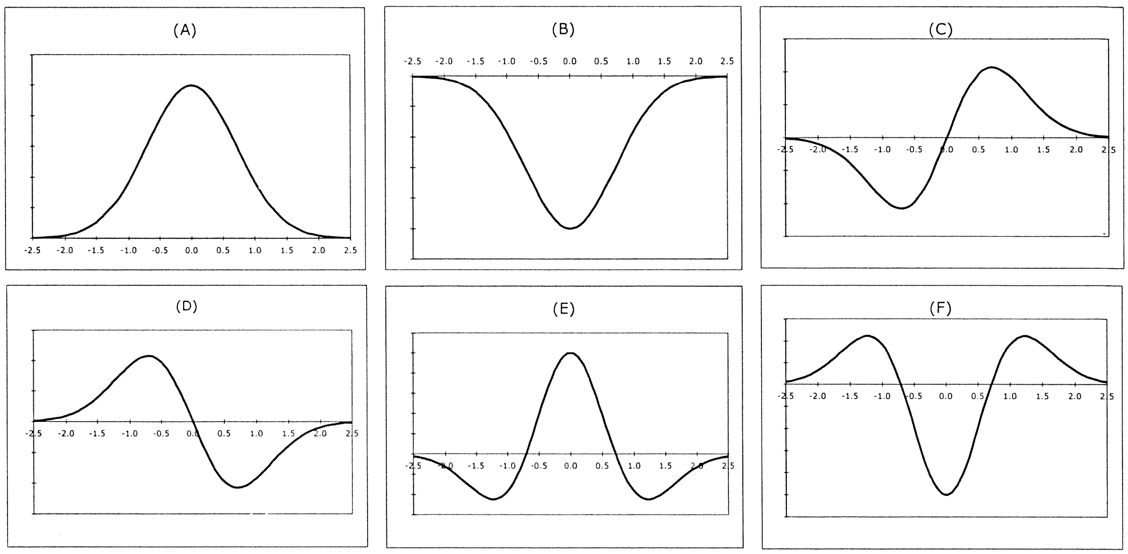
1 A Traveling Pulse [16 points]

A pulse travels to the right in the positive $x-$direction along a stretched string. In the figure below, (A) represents a snapshot picture of this pulse, $y(x, t_0)$, taken at time $t = t_0$. The units for x on the horizontal axis are the same in all pictures. The vertical axes have not been given numerical values; you may assume that their units are arbitrary.



1.1 (4 points)

Which graph represents the transverse velocity of the pulse, $\partial y(x,t)/\partial t|_{t=t_0}$? $\boxed{C}$

$$\frac{\partial y}{\partial t} = -v f'$$

1.2 (4 points)

Challenge:

Which graph represents the transverse acceleration of the pulse, $\partial^2 y(x,t)/\partial t^2|_{t=t_0}$? $\boxed{F}$

$$\frac{\partial^2 y}{\partial t^2} = +v^2 f''$$

1.3 (4 points)

A long time later, at $t = t_1$, the pulse has been reflected at a *free* boundary and is passing the same location again but moving to the left in the negative x-direction.

Which graph represents a snapshot of the reflected returning pulse at $t = t_1$, $y(x, t_1)$? $\boxed{A}$

1.4 (4 points)

Which graph represents the transverse velocity of the reflected pulse, $\partial y(x,t)/\partial t|_{t=t_1}$? $\boxed{D}$

2 Explicit expression for a reflected pulse [20 points]

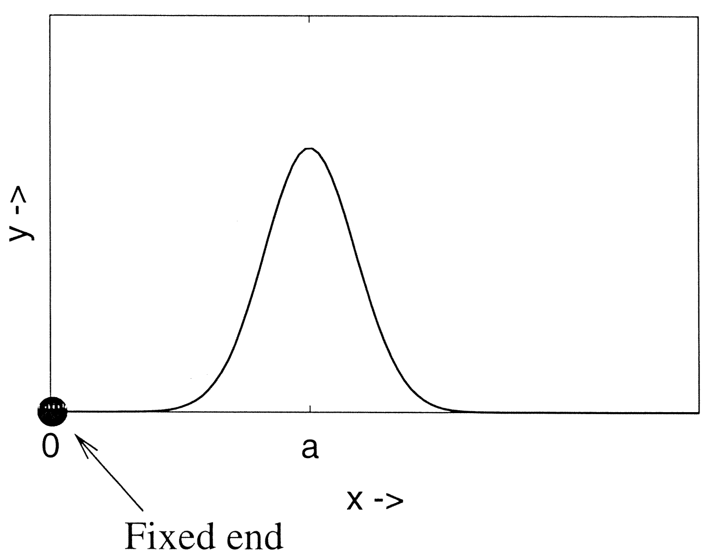
A long stretched string is oriented along the x-axis with its left end fixed at $x = 0$. The wave speed on the string is v. The solution to the wave equation for the string has the form

$$y(x,t) = f(x - vt) + g(x + vt)$$

with

$$g(x + vt) = Ae^{-(x+vt-a)^2/b^2}$$

The picture below shows the string at $t = 0$. The pulse shown is moving to the left.



2.1 (7 points)

Use the boundary condition at $x = 0$ to find an equation relating the functions $f(u)$ and $g(u)$. Do *not* insert the specific functions for f and g yet.

$0 = f(u) + g(-u)$ $\quad u = -vt$

$f(u) = -g(-u)$

2.2 (7 points)

Find $f(u)$.

$$g(-u) = Ae^{-(-u-a)^2/b^2} = Ae^{-(u+a)^2/b^2}$$

$$\boxed{f(u) = -Ae^{-(u+a)^2/b^2}}$$

2.3 (6 points)

Write down $y(x,t)$ in terms of only A, v, a, b, x, t, and fundamental constants, and show that $y(x,t)$ satisfies your boundary condition from 2.1.

for f: $u \longrightarrow x - vt$

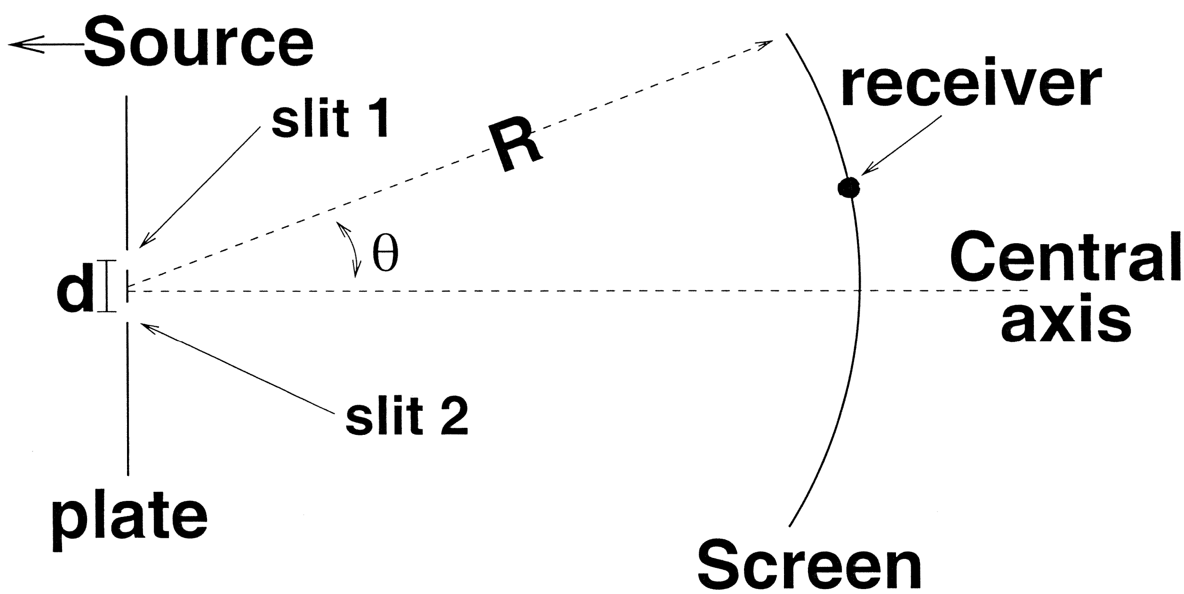
makes f travel to the right

$$\boxed{y(x,t) = -Ae^{-(x-vt+a)^2/b^2} + Ae^{-(x+vt-a)^2/b^2}}$$

at $x=0$: $y(0,t) = -Ae^{-(-vt+a)^2/b^2} + Ae^{-(vt-a)^2/b^2} = 0$

3 Double Slit Interference [19 points]

A beam of microwave plane waves incident from the left directly along the central axis in the figure below encounters a metal plate with one or more narrow slits. The receiver, always at distance R from the slit plate on the side opposite the generator of the beam, can be moved in an arc to vary the angle θ to the central axis. (See figure below.)



When there is only one slit open in the plate, the intensity measured on the other side of the slit by a receiver placed on the central axis of the arrangement ($\theta = 0$) at a distance R is I_0.

You now add a second slit of the same size. The two slits are $d = 5.0$ cm apart from each other, arranged symmetrically about the central axis, as shown, with $d << R$.

3.1 (5 points)

In terms of I_0, what is the intensity measured by the receiver when it is placed on the central axis ($\theta = 0$) with both slits open?

$\Delta\varphi = 0$

$\boxed{I = 4I_0}$

3.2 (8 points)

You move your receiver to $\theta = 15°$ and find that this is the *first* angle at which the intensity decreases to zero. What is the wavelength, λ, of the microwaves?

$$\Delta\varphi = \pi$$

$$k\Delta R = kd\sin\Theta = \pi \qquad k = \frac{2\pi}{\lambda}$$

$$\frac{2d}{\lambda}\sin\Theta = 1$$

$$\boxed{\lambda = 2d\sin 15° = 2.59\text{ cm}}$$

3.3 (6 points)

You move the detector back to the central axis ($\theta = 0$) and you cover one of the two slits with a piece of polyethylene of thickness b and index of refraction $n = 1.6$. In terms of n, b, λ, and mathematical constants as needed, what is the difference in phase $\Delta\phi$ between the microwaves arriving at the detector from the two slits?

$$\Delta\phi = k_{Poly}b - k_{Air}b$$

$$= (n-1)k_{air}b$$

$$\boxed{\Delta\phi = (n-1)\frac{2\pi}{\lambda}b}$$

$$\lambda_{Poly} = \frac{\lambda_{air}}{n}$$

$$k_{Poly} = n\,k_{air}$$

4 Standing electromagnetic waves [20 points]

The electric field of an electromagnetic standing wave of angular frequency ω in vacuum is given by

$$\vec{E}(x,t) = E_0 \sin(kx)\cos(\omega t)\ \hat{y}$$

4.1 (3 points)

Does $\vec{E}(x,t)$ have a node or an antinode at $x = 0$?

node

4.2 (7 points)

$\vec{E}(x,t)$ can be written in the form of the general solution of the wave equation,

$$\vec{E}(x,t) = \vec{E}_1(x,t) + \vec{E}_2(x,t),$$

where $\vec{E}_1(x,t) \equiv f(x-ct)\hat{y}$ and $\vec{E}_2(x,t) \equiv g(x+ct)\hat{y}$. Give explicit formulas for $\vec{E}_1(x,t)$ and $\vec{E}_2(x,t)$ in terms of *only* E_0, c, k, ω, and appropriate unit vectors as needed.

Hint: $\cos(\alpha)\sin(\beta) = (\sin(\alpha+\beta) - \sin(\alpha-\beta))/2$.

$$\vec{E}(x,t) = E_0 \cos(\omega t) \sin(kx)\ \hat{y}$$

$\beta = kx$

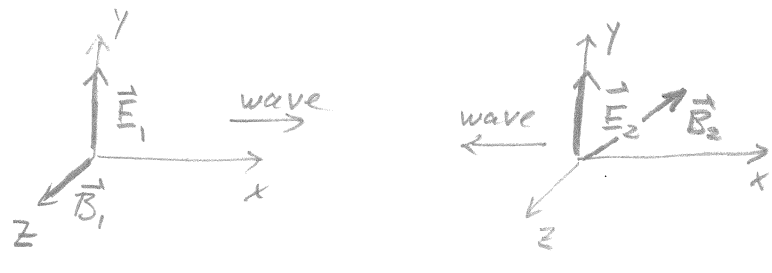
$\alpha = \omega t$

$$E_0 \cos(\omega t)\sin(kx) = \underbrace{\frac{E_0}{2}\sin(\omega t + kx)}_{E_2} - \underbrace{\frac{E_0}{2}\sin(\omega t - kx)}_{E_1}$$

$$\boxed{\begin{aligned}\vec{E}_1(x,t) &= \frac{E_0}{2}\sin(kx-\omega t)\ \hat{y} \\ E_2(x,t) &= \frac{E_0}{2}\sin(kx+\omega t)\ \hat{y}\end{aligned}}$$

4.3 (6 points)

Find explicit formulas for the magnetic fields $\vec{B}_1(x,t)$ and $\vec{B}_2(x,t)$ associated with the electric fields $E_1(x,t)$ and $E_2(x,t)$ in terms of *only* E_0, c, k, ω, and appropriate unit vectors as needed.



$$\boxed{\begin{aligned}\vec{B}_1(x,t) &= \frac{E_0}{2c}\sin(kx-\omega t)\,\hat{z}\\ \vec{B}_2(x,t) &= -\frac{E_0}{2c}\sin(kx+\omega t)\,\hat{z}\end{aligned}}$$

4.4 (4 points)

What is the value at $x = 0$ of the magnetic field associated with the standing wave, $\vec{B}(0,t)$? <u>Also</u>, the magnetic field has either a node or an antinode at this point. Which is it?

at $x = 0$

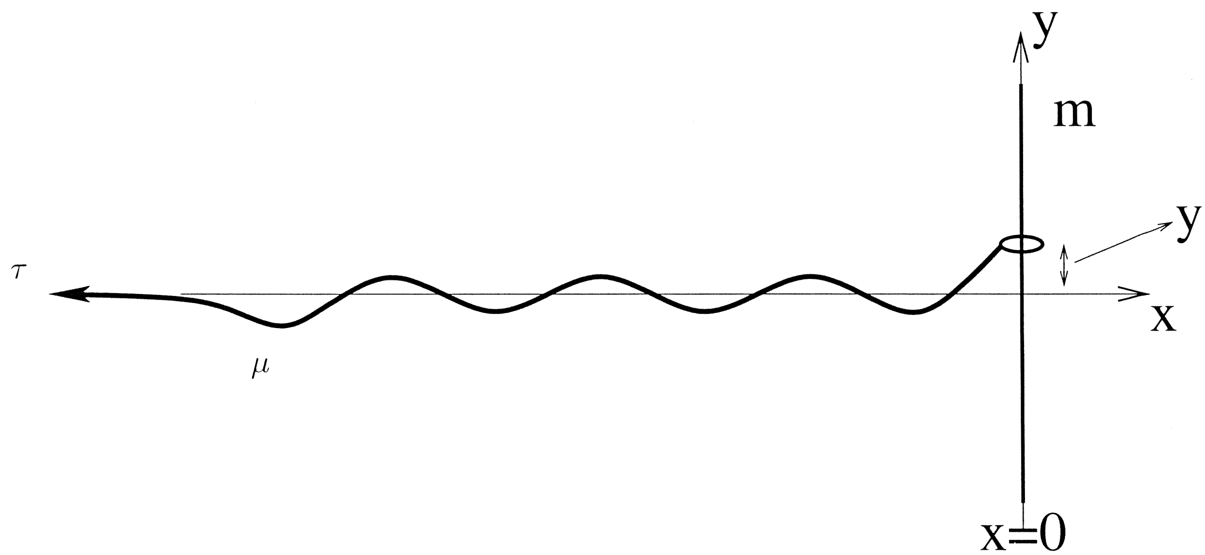
$$\vec{B}(0,t) = \vec{B}_1(0,t) + \vec{B}_2(0,t) = \left(\frac{E_0}{2c}\sin(-\omega t) - \frac{E_0}{2c}\sin(\omega t)\right)\hat{z}$$

$$\boxed{\vec{B}(0,t) = -\frac{E_0}{c}\sin(\omega t)\,\hat{z}}$$

<u>antinode</u>

5 The return of the ring [25 points]

A sinusoidal wave of complex amplitude $\underline{A}$ is sent from left to right along a long string oriented in the x-direction. The speed of transverse waves all along this string is c, the tension is τ. At $x = 0$ the string is attached to a vertical fixed rod by means of a frictionless ring of mass m. You may ignore the effects of gravity on the mass and on the string.



A solution to the wave equation for the string is given by

$$y(x \leq 0, t) = \Re\left[\underline{A}e^{-i\omega t}(e^{ikx} + \underline{B}e^{-ikx})\right]$$

5.1 (10 points)

Fill each box with the letter corresponding to the meaning of each of the following terms from the above expression.

$\underline{A}$: f

$\cdot\, e^{-i\omega t}$: i

e^{ikx}: c

$\underline{B}$: b

e^{-ikx}: e

~~(a) Radius of curvature~~

(b) Reflection coefficient

(c) Propagation phase of incident wave

~~(d) Transmission coefficient~~

(e) Propagation phase of reflected wave

(f) Complex amplitude of incident wave

~~(g) Vector potential~~

(i) Simple harmonic motion of each chunk at the same frequency

5.2 (8 points)

The boundary condition for the string at $x = 0$ is given by the equation of motion of the ring

$$m\frac{\partial^2 y}{\partial t^2}|_{x=0} = -\tau\frac{\partial y}{\partial x}|_{x=0}$$

Using this boundary condition, find the complex coefficient $\underline{B}$ in terms of *only* τ, m, k, c, and any numerical constants as needed.

$$y(x,t) = Re\left[\underline{A}e^{-i\omega t}\left(e^{ikx} + \underline{B}e^{-ikx}\right)\right]$$

$$\frac{\partial^2 y}{\partial t^2}\Big|_{x=0} = Re\left[-\omega^2 \underline{A}e^{-i\omega t}(1+\underline{B})\right]$$

$$\frac{\partial y}{\partial x}\Big|_{x=0} = Re\left[ik\underline{A}e^{-i\omega t}(1-\underline{B})\right]$$

$$-\omega^2(1+\underline{B})m = -\tau ik(1-\underline{B})$$

$$-m\omega^2 + ik\tau = \underline{B}(\omega^2 m + ik\tau) \qquad \omega^2 = k^2c^2$$

$$\boxed{\underline{B} = \frac{ik\tau - mk^2c^2}{ik\tau + mk^2c^2}}$$

5.3 (4 points)

What is $\underline{B}$ in the limit of $m \to \infty$ and $m \to 0$, respectively? What kind of termination of the string do these limits represent? **Hint:** Even if you have not completed 5.2, you may be able to answer this and the next problem based on physical intuition.

$m \to \infty \qquad \underline{B} \to -1 \qquad$ fixed termination

$m \to 0 \qquad \underline{B} \to 1 \qquad$ free "

5.4 (3 points)

What is the numerical value of $|\underline{B}|$? Explain your result (one or two sentences) in terms of conservation of energy.

$$|\underline{B}| = \sqrt{\underline{B}\,\underline{B}^*} = \left(\frac{ik\tau - m\omega^2}{ik\tau + m\omega^2} \cdot \frac{-ik\tau - m\omega^2}{-ik\tau + m\omega^2}\right)^{1/2}$$

$$= \left((-1)(-1)\right)^{1/2} = \underline{1}$$

Since there is no dissipation (friction), no energy is lost at the termination, and the reflected wave has the same amplitude as the incident wave. $\underline{B}$ shifts the phase, though.