Cornell University

Department of Physics

Phys 214 November 15, 2004

Waves, Optics, and Particles, Fall 2004

Homework Assignment # 10

(Due Thursday, November 18 at 5:00pm *sharp*.)

Agenda and readings for the week of November 17:

Skills to be mastered:

- Be able to compute potential, kinetic and total energy densities and power for a string or for a sound wave, given the wave solution
- Understand conservation of energy in wave motion
- Be able to compute electric and magnetic energy densities in an electromagnetic wave
- Be able to compute power flux for a plane E&M wave
- Understand the concept of intensity of an E&M wave and its relation to the power flux and the complex amplitude of the wave.

Lectures and Readings:

Readings marked YF are from the text Young and Freedman, *University Physics*, 11th edition.

- Lec 23, 11/16 (Tue): Introduction to Quantum Mechanics, G.P. Thomson experiment.
  Readings: YF 39.1.
- Lec 24, 11/18 (Thu): Analysis of G.P. Thomson experiment, measurement of $\hbar$, de Broglie hypothesis.
  Readings: YF 39.2.
- Lec 25, 11/23 (Tue): Heisenberg Uncertainty Principle.
  Readings: YF 39.3.
- Lec 26, 11/30 (Tue): Particles in a box; Schrödinger's equation.
  Readings: YF 39.5, 40.1–40.3.
- Lec 27, 12/02 (Thu): Three Nobel Ideas.
  Readings: LN "Feynman Diagrams . . . ," Secs. 3.4, 3.5, 4, 5.

1 Energy in a Standing Sound Wave

Consider a standing sound wave in a thin tube of cross-sectional area A with two open ends at $x = 0$ and $x = L$:

$$s(x,t) = S_0 \cos(kx)\cos(\omega t).$$

The bulk modulus and density are B and ρ_0, respectively.

(a) Find the kinetic energy density, $ke(x,t)$, and the potential energy density, $pe(x,t)$. Is it true that $ke(x,t) = pe(x,t)$?

(b) What is the total energy density, $e(x,t)$? Does it depend on time?

(c) What is the total energy in the standing wave? Does it depend on time?

(d) Find the power $P(x,t)$ (the rate at which energy flows past point x). [Careful: the quantity $\mathcal{P} \equiv -B\frac{\partial s}{\partial x}\frac{\partial s}{\partial t}$ is the instantaneous power *per unit area*, so the power is $P = \mathcal{P}A$.]

(e) Let $\mathcal{P}$ represent the instantaneous power per unit area. Show that the equation

$$\frac{\partial e}{\partial t} = -\frac{\partial \mathcal{P}}{\partial x}$$

holds at all x and t. What is the physical meaning of this equation?

(f) At what times t is the total kinetic energy of the sound wave a maximum?

(g) At what times t is the total potential energy of the sound wave a maximum?

(h) What is the power as a function of time at a node?

(i) What is the power as a function of time at an antinode?

(j) What are the implications of your answers to (h) and (i) for energy transport in the wave?

2 Intensity of Light

Consider a plane, linearly polarized electromagnetic wave in a dielectric described by ϵ and μ. Using complex representation, the fields in the wave are given by

$$\begin{aligned}\vec{E}(z,t) &= \Re(\underline{E}_0\, e^{ikz}e^{-i\omega t})\,\hat{\mathbf{x}},\\ \vec{B}(z,t) &= \frac{1}{c}\Re(\underline{E}_0\, e^{ikz}e^{-i\omega t})\,\hat{\mathbf{y}},\end{aligned} \tag{1}$$

where $c = 1/\sqrt{\epsilon\mu}$.

(a) Compute the densities of electric field energy u_{E} and magnetic field energy u_{B} as functions of z and t. Simplify your answers so that they do not contain any complex numbers.

(b) Compute the total energy density $u(z,t) = u_{\mathrm{E}}(z,t) + u_{\mathrm{B}}(z,t)$.

(c) Compute the power flux vector (usually called “Poynting vector” in eletromagnetic theory) $\vec{S}(z,t) \equiv \frac{1}{\mu}\vec{E} \times \vec{B}$. Which direction is energy flowing in? Express the z-component of the Poynting vector, $S_z(z,t) \equiv \vec{S}(z,t)\cdot\hat{\mathbf{z}}$, in a form that does not contain any complex numbers.

(d) Show that $S_z(z,t)$ and $u(z,t)$ satisfy a continuity equation of the expected form.

3 Energy in Pulses I

A pulse moving in the $+x$ direction on a string has shape at $t = 0$ given by $y(x, t = 0) = f(x)$.

(a) Show that the total energy of the pulse is

$$E_0 = \int_{-\infty}^{+\infty} \tau [f'(x)]^2 dx$$

How much of the energy is kinetic and how much is potential?

(b) Suppose a pulse on the same string had the same shape but the amplitude was scaled by a factor a. In other words, at $t = 0$ the shape is given by $y(x, t = 0) = af(x)$. What is the energy of this pulse in terms of E_0?

(c) Suppose a pulse on the same string had the same shape and amplitude as the original pulse but was stretched horizontally by a factor b. In other words, at $t = 0$ the shape is given by $y(x, t = 0) = f(x/b)$. What is the energy of this pulse in terms of E_0?

4 Energy in Pulses II

Two half-infinite strings are connected at $x = 0$. The wave speed for $x < 0$ is c_1 and the wave speed for $x > 0$ is c_2. A pulse is coming from the left toward $x = 0$. The shape of the incident pulse at $t = 0$ is $y(x, t = 0) = f(x)$. The total energy of the incident pulse is

$$E_\mathrm{i} \equiv \int_{-\infty}^{+\infty} \tau [f'(x)]^2 dx$$

After a while, reflected and transmitted pulses are moving away from $x = 0$.

(a) Compute the total energy E_r of the reflected pulse. Express your answer in terms of E_i, c_1, and c_2. [Hint: use what you learned in the previous problem.]

(b) Compute the total energy E_t of the transmitted pulse. Express your answer in terms of E_i, c_1, and c_2. Verify conservation of energy.

5 The propagation decay factor $p(r)$

In class, we talked about a propagation decay factor $p(r)$ that describes how the amplitude of a wave decreases with distance r from the source. We wrote the complex amplitude of a wave as

$$\underline{Q}(r) = p(r)e^{ikr}$$

where e^{ikr} is the phase factor for propagating a distance r.

(a) Suppose a point source radiates waves isotropically (equally in all directions). How do you expect the rate of energy flow through a spherical surface at distance r_1 from the source to compare to the rate of energy flow through a spherical surface at distance $r_2 > r_1$?

(b) The intensity I is the rate of energy flow through a surface per unit area (power per unit area). Based on your answer to (a), how does I depend on r?

(c) The intensity is proportional to the square of the amplitude of the wave: $I(r) \propto |\underline{Q}(r)|^2$. How does $p(r)$ depend on r?

(d) Challenge: For waves traveling in two dimensions (such as vibrations traveling away from a point source on a thin membrane or sheet), how do you expect $p(r)$ to depend on the distance r from the source? Explain.

(e) For waves traveling in one dimension (such as transverse waves traveling away from a point source on a string), how do you expect $p(r)$ to depend on the distance r from the source? Explain.

6 Energy transport in a plucked guitar string

At $t = 0$, a guitar string fixed at both ends ($x = 0$ and at $x = L$) is plucked at its midpoint so that its initial shape is an isosceles triangle (Fig. 1) and its initial velocity is zero everywhere. The tension in the string is τ and the wavespeed is v. In Problem Set 7, we figured out how to predict the future shape of the string. Now we are going to look at energy transport.

(a) What is the total energy in the last quarter of the string (from $x = 3L/4$ to $x = L$) at $t = 0$?

(b) What is the total energy in the last quarter of the string (from $x = 3L/4$ to $x = L$) at $t = L/(2v)$?

(c) Sketch a graph of the power P as a function of time at the point $x = 3L/4$ between $t = 0$ and $t = L/(2v)$.

(d) Integrate $P(x = 3L/4, t)$ with respect to time from $t = 0$ to $t = L/(2v)$ to find the energy that enters the last quarter of the string during that time interval. Is your answer consistent with your answers to (a) and (b)? Explain.

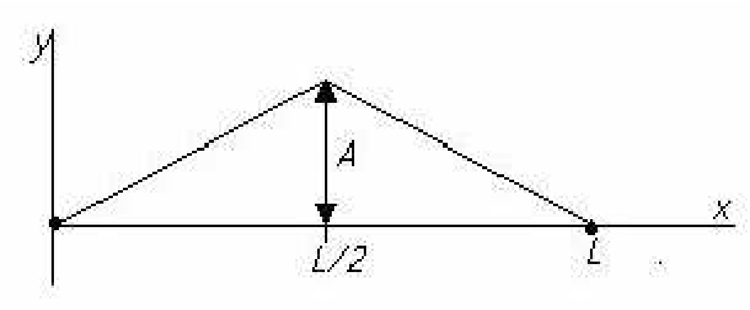

Figure 1: Initial shape of a plucked guitar string.