1. Energy in a Standing Sound Wave

We are given a sound wave in an open tube;

$$s(x,t) = S_0 \cos(kx)\cos(\omega t),$$

with bulk modulus B and density ρ_0.

ⓐ The kinetic energy density is given by

$$ke(x,t) = \frac{1}{2}\rho_0 \left(\frac{\partial s}{\partial t}\right)^2 = \frac{1}{2}\rho_0\omega^2 S_0^2 \cos^2(kx)\sin^2(\omega t).$$

The potential energy density is

$$pe(x,t) = \frac{1}{2}B\left(\frac{\partial s}{\partial x}\right)^2 = \frac{1}{2}Bk^2 S_0^2 \sin^2(kx)\cos^2(\omega t).$$

Since $c = \sqrt{B/\rho_0} = \omega/k$, so that $Bk^2 = \rho_0\omega^2$, we see that the potential and kinetic energies are <u>not the same</u>, nor do we expect them to be.

ⓑ The total energy density,

$$\begin{aligned} e(x,t) &= ke(x,t) + pe(x,t) \\ &= \frac{1}{2}Bk^2S_0^2\left(\cos^2(kx)\sin^2(\omega t) + \sin^2(kx)\cos^2(\omega t)\right). \end{aligned}$$

We can rewrite $\cos^2(\omega t) = 1 - \sin^2(\omega t)$ to see that indeed $e(x,t)$ <u>does</u> depend on time.

1. (c) The total energy in the standing wave is found by integrating the energy density over the length of the tube:

$$E(t) = A\int_0^L e(x,t)\,dx$$

$$= \frac{A}{2} S_0^2 B k^2 \left\{ \sin^2(\omega t) \int_0^L \cos^2(kx)\,dx + \cos^2(\omega t)\int_0^L \sin^2(kx)\,dx \right\}.$$

We use $\int_0^L \sin^2(kx)\,dx = \frac{L}{2} - \frac{\cos(2kx)}{4k}\Big|_0^L = \frac{L}{2}$, since $\cos(2kL) = \cos(0) = 1$ due to the open boundary. Likewise, $\int_0^L \cos^2(kx)\,dx = \frac{L}{2}$. Thus,

$$E(t) = \frac{A}{2} S_0^2 B k^2 \left(\frac{L}{2}\right)\left(\sin^2(\omega t) + \cos^2(\omega t)\right) = \frac{A}{4} S_0^2 B k^2 L.$$

(d) The power through a point at x is

$$P(x,t) = -B\left(\frac{\partial s}{\partial x}\right)\left(\frac{\partial s}{\partial t}\right)$$

$$= -B S_0^2 k\omega \cos(kx)\sin(kx)\cos(\omega t)\sin(\omega t)$$

$$= -\frac{B}{4} S_0^2 k\omega \cdot \sin(2kx)\sin(2\omega t).$$

(e) We confirm,

$$\frac{\partial e}{\partial t} = \frac{1}{2} B k^2 S_0^2 \left[\omega \cos^2(kx)\cdot 2\cos(\omega t)\sin(\omega t) - \omega \sin^2(kx)\cdot 2\sin(\omega t)\cos(\omega t)\right]$$

$$= \frac{1}{2} B k^2 \omega S_0^2 \left(\cos^2(kx) - \sin^2(kx)\right)\left(2\cos(\omega t)\sin(\omega t)\right)$$

$$= \frac{1}{2} B k^2 \omega S_0^2 \cos(2kx)\sin(2\omega t)$$

1. (e) (cont'd) Further,

$$\frac{\partial P}{\partial x} = -\frac{B}{4} S_0^2 k\omega \left[2k\cos(2kx)\right] \sin(2\omega t) = -\frac{\partial e}{\partial t}.$$

So this equation holds everywhere, always. This is the continuity equation, stating that energy is neither created, nor destroyed, nor teleported.

(f) We integrate the kinetic energy density,

$$KE(t) = A\int_0^L ke(x,t)\,dx = \frac{AS_0^2}{4} Bk^2 L \cdot \sin^2(\omega t).$$

This is maximum for $t = \frac{2n+1}{2}\frac{\pi}{\omega}$, $n = 0, 1, 2, \ldots$.

(g) The total potential energy, likewise,

$$PE(t) = A\int_0^L pe(x,t)\,dx = \frac{AS_0^2}{4} Bk^2 L \cdot \cos^2(\omega t),$$

which is maximal at $t = \frac{n\pi}{\omega}$, $n = 0, 1, 2, \ldots$.

(h) The displacement $s(x,t) = S_0\cos(kx)\sin(\omega t)$ has nodes at $kx = \frac{2n+1}{2}\pi$. Plugging this into $P(x,t)$,

$$P\left(\frac{2n+1}{2k}\pi, t\right) = -\frac{B}{4} S_0^2 k\omega \sin([2n+1]\pi)\sin(2\omega t) = 0.$$

(i) The antinodes occur at $kx = n\pi$. Substituting,

$$P\left(\frac{n\pi}{k}, t\right) = -\frac{B}{4} S_0^2 k\omega \sin([2n]\pi)\sin(2\omega t) = 0.$$

(j) Energy is transported between nodes and antinodes, but no energy ever crosses them.

2. Intensity of Light

We have a travelling E-M wave:

$$\vec{E}(z,t) = \text{Re}\left[\underline{E_0}\, e^{ikz} e^{-i\omega t}\right]\hat{x} = E_0 \cos(kz-\omega t+\varphi_0)\,\hat{x}$$

$$\vec{B}(z,t) = \frac{1}{c}\text{Re}\left[\underline{E_0}\, e^{ikz} e^{-i\omega t}\right]\hat{y} = \frac{E_0}{c}\cos(kz-\omega t+\varphi_0)\,\hat{y},$$

where $\underline{E_0} = E_0 e^{i\varphi_0}$ is broken into modulus and phase

(a) We find the field energy densities,

$$u_E = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\varepsilon E_0^2 \cos^2(kz-\omega t+\varphi_0)$$

$$u_B = \frac{1}{2\mu}B^2 = \frac{1}{2\mu}\frac{E_0^2}{c^2}\cos^2(kz-\omega t+\varphi_0) = u_E$$

Since $\varepsilon = \frac{1}{\mu c^2}$. We note the meaning of "E^2", since $\vec{E}$ is a vector in complex representation But $\vec{E}$ is defined as the Real part, so $E^2 = \vec{E}\cdot\vec{E} = \text{Re}(\cdots)\cdot\text{Re}(\cdots)$. In this case, it is best to use the trig form all the way. Note that $E^2 \neq EE^*$ — this is wrong!

(b) Since $u_E = u_B$, we have total energy

$$u(z,t) = 2u_E(z,t) = \varepsilon E_0^2\cos^2(kz-\omega t+\varphi_0).$$

(c) Again using the cosine form, and $\hat{x}\times\hat{y}=\hat{z}$, the Poynting vector

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B} = \frac{1}{\mu c}E_0^2\cos^2(kz-\omega t+\varphi_0)\,\hat{z}.$$

This tells us that energy flows in the $+\hat{z}$ direction, consistent with our travelling wave.

2. (d) We expect the continuity equation,

$$\frac{\partial u}{\partial t} + \frac{\partial S_z}{\partial z} = 0.$$

Indeed,

$$\frac{\partial u}{\partial t} = \varepsilon E_0^2 \left(2\omega \cos(kz-\omega t+\varphi_0)\sin(kz-\omega t+\varphi_0)\right),$$

$$\frac{\partial S_z}{\partial z} = \frac{1}{\mu c} E_0^2 \left(-2k \cos(kz-\omega t+\varphi_0)\sin(kz-\omega t+\varphi_0)\right).$$

Adding these gives

$$\frac{\partial u}{\partial t} + \frac{\partial S_z}{\partial z} = \left(\varepsilon\omega - \frac{k}{\mu c}\right) E_0^2 \sin\left(2[kz-\omega t+\varphi_0]\right) = 0$$

Since

$$\varepsilon\omega - \frac{k}{\mu c} = \varepsilon\omega - \frac{\omega}{\mu c^2} = 0.$$

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3. Energy in Pulses I

We have a pulse in the $+x$ direction with initial shape $y(x,t=0) = f(x)$. Thus,

$$y(x,t) = f(x-vt).$$

(a) We know for a pulse that kinetic and potential energy densities are equal:

$$e(x,t) = ke(x,t) + pe(x,t) = \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2.$$

But $\frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x}$ so that $\mu v^2 = \tau$ gives

$$e(x,t) = \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{2}\mu v^2\left(\frac{\partial y}{\partial x}\right)^2 = \tau\left(\frac{\partial y}{\partial x}\right)^2 = \tau\left(f'(x-vt)\right)^2.$$

3. ⓐ (cont'd) Now we find the total energy by integrating over the infinite length of the string:

$$E_0 = \int_{-\infty}^{\infty} e(x,t)\,dx = \int_{-\infty}^{\infty} \tau\left(f'(x-vt)\right)^2 dx.$$

We substitute $u = x - vt$, $du = dx$ so that

$$E_0 = \int_{-\infty}^{+\infty} \tau\left[f'(u)\right]^2 du = \int_{-\infty}^{\infty} \tau\left[f'(x)\right]^2 dx$$

since u and x are simply dummy variables.

ⓑ If instead $y(x,t) = af(x-vt)$, we find

$$E = \int_{-\infty}^{\infty} \tau\left[\frac{\partial}{\partial x} af(x-vt)\right]^2 \cancel{dx} = \int_{-\infty}^{\infty} \tau\left[a\frac{\partial}{\partial x} f(x-vt)\right]^2 dx$$

$$= \int_{-\infty}^{\infty} \tau a^2\left[f'(x-vt)\right]^2 dx = a^2 E_0.$$

ⓒ If the wave is horizontally stretched by a factor of b, so that $y(x,t) = f\left(\frac{x-vt}{b}\right)$, we get

$$E = \int_{-\infty}^{\infty} \tau\left[\frac{\partial}{\partial x} f\left(\frac{x-vt}{b}\right)\right]^2 dx = \int_{-\infty}^{\infty} \tau\left[\frac{\partial}{\partial x} f\left(\frac{x}{b}\right)\right]^2 dx,$$

where we substitute $x - vt \to x$ as above. We use the chain rule, $\frac{\partial}{\partial x} f(x/b) = \frac{1}{b} f'(x/b)$, so that

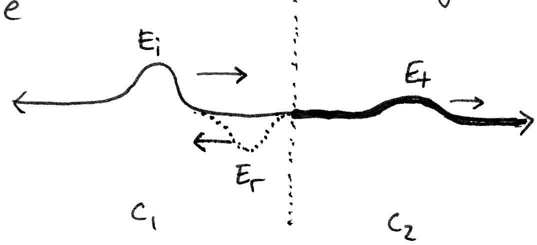
$$E = \int_{-\infty}^{\infty} \tau \frac{1}{b^2} f'\left(\frac{x}{b}\right) dx.$$

But we must change $u = \frac{x}{b}$, $du = \frac{dx}{b}$, so that

$$E = \int_{-\infty}^{\infty} \tau \frac{1}{b^2} f'(u)\, b\,du = \int_{-\infty}^{\infty} \frac{\tau}{b} f'(x)\,dx = \frac{1}{b} E_0.$$

4. Energy in Pulses II

We now look at energy transport at an interface



*note, the transmitted wave should be narrower if $c_2 < c_1$, as is suggested in the picture.

(a) We write the equation $y_r(x,t)$ for the reflected pulse as

$$y_r(x,t) = R\, f(x+vt)$$

where R is the reflection coefficient. We needn't worry that it can interfere with the incoming wave — we will wait until they have passed one another. Now, from (3b),

$$E_r = R^2 E_i .$$

We ignore the fact that it is $f(x+vt)$, since it is irrel v ant. R is given by

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{T/c_1 - T/c_2}{T/c_1 + T/c_2} = \frac{c_2 - c_1}{c_2 + c_1}$$

where we use $Z = T/c$ for a string. Thus,

$$E_r = R^2 E_i = \frac{c_2^2 - 2c_1c_2 + c_1^2}{(c_1+c_2)^2} E_i .$$

4. (b) We use a similar analysis for the transmission, except that the condition

$$y_1(0,t) = y_2(0,t)$$

makes the wave stretch/shrink horizontally by a factor of c_1/c_2:

$$y_t(x,t) = T f\left(\frac{c_1}{c_2}(x-vt)\right).$$

We combine (3a) & (b) so that the energy multiplies by T^2 and $\frac{c_1}{c_2}$:

$$E_t = T^2 \frac{c_1}{c_2} E_i = \frac{4c_1c_2}{(c_1+c_2)^2} E_i .$$

where $T = \frac{2z_1}{z_1+z_2} = \frac{2\tau/c_1}{\tau/c_1+\tau/c_2} = \frac{2c_2}{c_1+c_2}$.

We verify conservation of energy, since

$$E_r + E_t = \left[\frac{c_1^2 - 2c_1c_2 + c_2^2}{(c_1+c_2)^2} + \frac{4c_1c_2}{(c_1+c_2)^2}\right] E_i$$

$$= \frac{c_1^2 + 2c_1c_2 + c_2^2}{(c_1+c_2)^2} E_i = E_i .$$

So all the incident energy goes into either the transmitted or reflected wave.

5. The propagation decay factor $p(r)$

We write the complex amplitude

$$\underline{Q}(r) = p(r)\, e^{ikr}.$$

(a) We expect the total rate of flow through any surface surrounding the point source to be the same,

$$\oint_{r_1} \mathbf{I}(r) \cdot d\mathbf{a} = \oint_{r_2} \mathbf{I}(r) \cdot d\mathbf{a}.$$

(b) The flux through a sphere is just $4\pi r^2 I(r)$, so that we need

$$4\pi r_1^2 I(r_1) = 4\pi r_2^2 I(r_2) = \text{constant},$$

or

$$I(r) \sim 1/r^2.$$

(c) If $I(r) \propto |\underline{Q}(r)|^2$ then we have

$$\underline{Q}(r) \sim 1/r.$$

Thus,

$$p(r) \sim 1/r.$$

(d) In two dimensions, $2\pi r I(r) = \text{constant}$ so

$$I(r) \sim 1/r.$$

Thus,

$$p(r) \sim 1/\sqrt{r}.$$

(e) In one-dimensional propagation, $2 I(r) = \text{constant}$. This implies $p(r) \sim 1$ is independent of r!

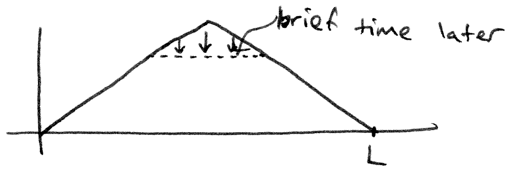
6. Energy transport in a plucked guitar string

We refer to PS7 for the solutions:

(a) The total energy is given by the density

$$e = \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2, \qquad \mu = \tau/v^2$$

In the last quarter of the string there is no motion:



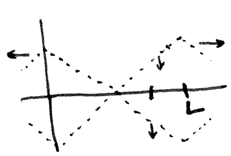
Thus all the energy is potential:

$$e = \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}\tau\left(\frac{2A}{L}\right)^2 = \frac{2A^2\tau}{L^2}.$$

We integrate

$$E = \int_{\frac{3}{4}L}^{L} e(x)\,dx = \frac{2A^2\tau}{L^2}\left(\frac{1}{4}L\right) = \frac{A^2\tau}{2L}.$$

(b) At time $t = \frac{L}{2v}$ we have the two travelling waves which perfectly cancel. Each has a slope $\frac{A}{L}$ so that $\frac{\partial y}{\partial t} = v\frac{A}{L}$ for each one. Thus,



$$e = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}(\tau/v^2)\cdot\left(2v\frac{A}{L}\right)^2 = \frac{2A^2\tau}{L^2}$$

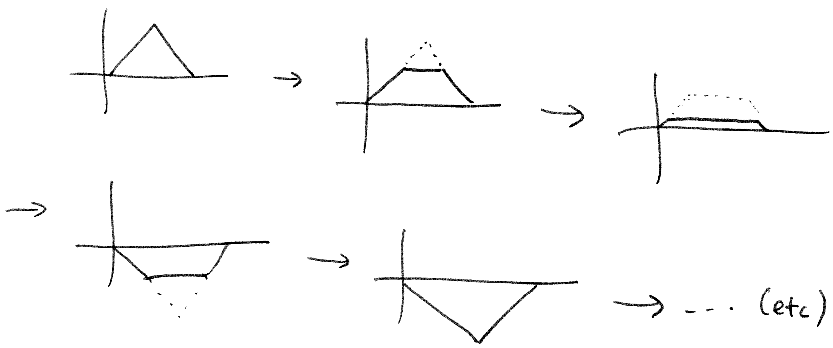
We again integrate this constant to find

$$E = \int_{\frac{3}{4}L}^{L} e(x)\,dx = \frac{2A^2\tau}{L^2}\left(\frac{1}{4}L\right) = \frac{A^2\tau}{2L}.$$

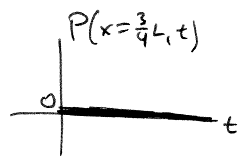
6. ⓒ The power at any point in the string is given by

$$P(x,t) = -\tau\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right).$$

But in our case,



we see that only one of these is ever non-zero: There is only (vertical) $\frac{\partial y}{\partial t}$ if the slope $\frac{\partial y}{\partial x} = 0$, and vice-versa. Thus, $P(x,t) = 0$.



ⓓ We integrate

$$\Delta E = \int_0^{L/2v} P(x=\tfrac{3}{4}L,t)\,dt = 0.$$

This is consistent because no energy enters or leaves this part of the string. (ever).