Cornell University

Department of Physics

Phys 214 November 22, 2004

Waves, Optics, and Particles, Fall 2004

Homework Assignment # 12

(Due Thursday, December 2 at 5:00pm *sharp*.)

Skills to be mastered:

- Understand and apply Heisenberg's uncertainty principle
- Understand the motion of a particle in a box from quantum mechanical point of view
- Be able to solve Schrödinger's equation for one-dimensional problems
- Use Schrödinger's equation and Feynman's "sum over histories" approach to study particle motion in one-dimensional potentials

Lectures and Readings:

- Lec 25, 11/23 (Tue): Heisenberg Uncertainty Principle.
  Readings: YF 39.3.
- Lec 26, 11/30 (Tue): Particles in a box; Schrödinger's equation.
  Readings: YF 39.5, 40.1–40.3; LN "Quantum III: Particle in a box . . . ," Secs. 1.1–1.6; LN "Quantum IV: Scattering Theory," Sec. 4.
- Lec 27, 12/02 (Thu): Three Nobel Ideas.
  Readings: LN "Feynman Diagrams . . . ," Secs. 3.4, 3.5, 4, 5.

1 Dye Laser

An electron in a long, organic molecule used in a dye laser behaves approximately like a particle in a box with width 4.18 nm.

(a) Sketch the wave functions and give the energies (in eV) for the three lowest energy levels of the electron.

(b) What is the energy (in eV) of the photon emitted when the electron makes a transition from the first excited state ($n = 2$) to the ground state ($n = 1$)? Compare to the range of energies of photons in the visible range (about 1.8-3.1 eV).

(c) What is the energy (in eV) of the photon emitted when the electron makes a transition from the second excited state ($n = 3$) to the ground state?

2 Uncertainty Principle and the Particle in a Box

An electron (mass m) is confined to a one-dimensional region $0 \leq x \leq L$. In the ground state, its wave function is

$$\psi_n(x) = A\sin(k_n x) = \frac{A}{2i}(e^{ik_n x} - e^{-ik_n x})$$

(a) What are the possible values of k_n?

(b) In the ground state ($n = 1$), what are the possible values of the electron's momentum p_x? Are these values equally likely? Explain.

(c) Based on your answer to (b), what is the uncertainty* in the momentum Δp_x?

(d) What is the uncertainty* in position Δx?. Calculate the product $\Delta x \Delta p_x$ and interpret your answer.

*Note: we have not defined precisely what we mean by "uncertainty." For Problem 2, use the following crude definition:

$$\Delta Q = \frac{1}{2}(\text{maximum possible value of } Q - \text{minimum possible value of } Q)$$

Without a precise definition of uncertainty, we can only expect the Heisenberg Uncertainty Principle (HUP) to give us order-or-magnitude estimates.

3 Size of the hydrogen atom

Consider a hydrogen atom. By the HUP, we know that the electron can't be located exactly at the nucleus. Nor will it always be at a fixed distance from the nucleus (as is assumed in the Bohr model). Let's try to make an order-of-magnitude estimate of the typical or average distance between the electron and the nucleus using the HUP. Call this distance r. [Note: since this is a very crude way to estimate r, don't sweat factors of 2 and the like, even if they seem to have a big effect on your numerical answer.]

(a) Assume that the electron is confined to a *one-dimensional* region $-r \leq x \leq r$. What is the minimum possible uncertainty Δp_x in its momentum x-component in terms of $\hbar$ and r?

(b) Since p_x can be positive or negative, the *average* value of p_x should be zero (there is no reason for it to be traveling in one direction or another on average). Given your answer to part (a), what is the *typical magnitude* of the electron's momentum? Use this typical momentum magnitude to find the typical kinetic energy of the electron in terms of just $\hbar$, r, and the mass of the electron m.

(c) Now we need to use something you learned in Physics 213. From the expression for the electric potential energy between two charges, what is the typical potential energy in terms of just the elementary charge e, the Coulomb constant k, and r?

(d) The electron in the atom will eventually find a probability distribution with a spread r that minimizes the total energy (KE + PE). What value for r (in Angstroms) gives the minimum total energy? Is this a reasonable order-of-magnitude estimate for the size of a hydrogen atom?

(e) What value do you get for the minimum energy (in electron-volts)? How does this compare with the ionization energy of the hydrogen atom?

4 Reflection and Transmission at Potential Step

A beam of particles of mass m and energy $E > U_0$ are moving in the $+x$ direction toward a potential energy step:

$$U(x < 0) = 0$$
$$U(x \geq 0) = U_0$$

Note: U_0 might be positive, negative, or zero.

(a) What is the momentum p of the incoming particles?

(b) Taking $k = p/\hbar$, we can write the wave function for the particles as

$$\psi_1(x) = e^{ik_1 x} + \underline{r}e^{-ik_1 x}, x < 0$$

and

$$\psi_2(x) = \underline{t}e^{ik_2 x}, x \geq 0$$

Explain the physical meanings of $\underline{r}$ and $\underline{t}$.

(c) What are the values of k_1 and k_2?

(d) Show that $\psi_1(x)$ and $\psi_2(x)$ are solutions of the Schrödinger equation (the equation of motion for the wave function $\psi(x)$)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x).$$

in the regions $x < 0$ and $x \geq 0$, repsectively.

(e) One boundary condition at $x = 0$ is that the wave function $\psi(x)$ must be continuous. Use this condition to find an equation relating $\underline{r}$ and $\underline{t}$.

(f) The other boundary condition at $x = 0$ is that the first derivative of the wave function $d\psi/dx$ must be continuous. Use this condition to find another equation relating $\underline{r}$ and $\underline{t}$. (You may leave k_1 and k_2 in your answer.)

(g) Find expressions for $\underline{r}$ and $\underline{t}$ in terms of k_1 and k_2.

(h) What is the probability that an incident particle is reflected? What would the probability be according to classical physics?

5 Reflection from a Quantum Pothole

A beam of particles of mass m and energy $E > 0$ are moving in the $+x$ direction toward a "quantum pothole":

$$U(x < 0) = 0$$
$$U(0 \leq x \leq a) = U_0 < 0$$
$$U(x > 0) = 0$$

The wave function for $x < 0$ is given by

$$\psi_1(x) = e^{ikx} + \underline{R}e^{-ikx}.$$

and the wave function for $x > a$ is given by

$$\psi_3(x) = \underline{T}e^{ikx}.$$

We will obtain the coefficients $\underline{R}$ and $\underline{T}$ using two different methods. [*Hint*: This problem is similar to problem 4 on problem set 8.]

5.1 Matching at the boundary

(a) The wave function for $0 \le x \le a$ takes the form

$$\psi_2(x) = \underline{C}e^{ik'x} + \underline{D}e^{-ik'x}$$

Determine k and k'.

(b) Write down the equations that should be satisfied by the wave functions at $x = 0$.

(c) Write down the equations that should be satisfied by the wave functions at $x = a$.

(d) Solve these equations to obtain $\underline{R}$.

5.2 Sums over histories

As for waves on a string, we can define "transmission" and "reflection" coefficients $\underline{t}$ and $\underline{r}$ at a boundary between regions with different potential energies and thus different wave vectors k_1 and k_2. In the previous problem, you dervied these coefficients for a wave going from region 1 to region 2 in terms of k_1 and k_2.

(a) What is the wave function at $x = 0$ of the wave that is reflected from the left edge of the pothole without entering into it?

(b) What is the wave function at $x = 0$ of the wave that enters the pothole, is reflected at $x = a$, and then escapes from the pothole at $x = 0$?

(b) Generalizing your result from parts (a) and (b), obtain the wave function at $x = 0$ of the wave reflected from the pothole by summing over all possible back-and-forth reflections within the pothole. Use this result to obtain $\underline{R}$.