HOMEWORK ASSIGNMENT #2

SOLUTIONS

1. Identifying solutions & general solutions :

The equation of motion is $m\frac{d^2x}{dt^2} = -K(x - x_{eq})$. We have to decide if the following are general solutions or not.

(a) $x(t) = C_1 + C_2 \cos\omega_1 t$

So, $\frac{d^2x}{dt^2} = -\omega_1^2 C_2 \cos\omega_1 t$

- Yes, this is a solution provided $\omega_1^2 = K/m$ & $C_1 = x_{eq}$.
- Number of adjustable parameters $= C_2$ (only one)
- So this is NOT a general solution.

(b) $x(t) = x_{eq} + A_1 \sin\omega_1 t + A_2 \cos\omega_1 t$

So, $\frac{d^2x(t)}{dt^2} = -\omega_1^2 A_1 \sin\omega_1 t - \omega_1^2 A_2 \cos\omega_1 t$

$= -\omega_1^2 (x(t) - x_{eq})$

- Yes, this is a solution provided $\omega_1^2 = K/m$.
- Number of adjustable parameters $= 2$ $(A_1 \& A_2)$.
- So this IS a general solution.

(c) $x(t) = x_{eq} + A_1 \sin\omega_1 t + A_2 \sin\omega_2 t$

$\frac{d^2x(t)}{dt^2} = -\omega_1^2 A_1 \sin\omega_1 t - \omega_2^2 A_2 \sin\omega_2 t$

- By inspection we can see that this will be a solution provided $\omega_1^2 = \omega_2^2 = K/m$.
- Then number of adjustable parameters $= 1$ $((A_1 + A_2))$.
- So this is NOT a general solution.

(d) $x(t) = x_{eq} + A_1 e^{i\omega_0 t} - A_2 e^{-i\omega_0 t}$

$$\frac{d^2 x(t)}{dt^2} = -\omega_0^2 \left(x(t) - x_{eq}\right)$$

- Yes, this is a solution provided $\omega_0^2 = K/m$.
- To find the number of adjustable parameter take the real part of $x(t)$. $\mathrm{Re}\,[x(t)] = x_{eq} + (A_1 - A_2)\cos\omega_0 t$. So there is only one adjustable parameter : $(A_1 - A_2)$!
- So this is NOT a general solution.

(e) $x(t) = x_{eq} + A\cos[\omega_0(t - t_0)]$

$$\frac{d^2 x(t)}{dt^2} = -\omega_0^2 A\cos[\omega_0(t - t_0)]$$

- Yes this is a solution provided $\omega_0^2 = K/m$.
- 2 adjustable parameters ; A & $-\omega_0 t_0 = \phi_0$, say (note that ω_0 is fixed by K & m. So it is t_0 that is adjustable, and consequently ϕ_0 too is adjustable).
- Yes, this IS a general solution.

(f) $x(t) = x_{eq} + A\tan\omega_0 t + B\sec\omega_0 t$

$$\frac{d^2 x}{dt^2} = \omega_0^2 \left\{ 2A\sec^2\omega_0 t\tan\omega_0 t + B\sec^3\omega_0 t + B\tan^2\omega_0 t\sec\omega_0 t \right\}$$

- This is NOT a solution for general A & B.
- In the trivial case when $A = B = 0$, $x(t) = x_{eq} =$ constant is a trivial solution.

(g) $x(t) = x_{eq} - A\cos(\phi_0 - \omega_0 t)$

$$\frac{d^2x}{dt^2} = \omega_0^2 x$$

- Yes, this is a solution if $\omega_0^2 = K/m$.
- 2 adjustable parameters – A & ϕ_0.
- So this IS a general solution.

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(2) Damped Oscillator :

(a) To write down the equation of motion, note that there are 2 forces acting on the body : the restoring force due to the spring $(-K(x-x_{eq}))$, and the drag force $\left(-bm\frac{dx}{dt}\right)$. Since the body moves just along the X-axis we replace $\vec{v}$ by its x-component, $\frac{dx}{dt}$.

So the equation of motion is :

$$\boxed{m\frac{d^2x}{dt^2} = -K(x-x_{eq}) - bm\frac{dx}{dt}} \qquad (1)$$

(b) Real general solution using complex representation :

Let $x = \text{Re}\left\{x_{eq} + \underline{A}\, e^{i\underline{\omega}t}\right\}$ where both $\underline{A} = Ae^{i\phi_0}$, and $\underline{\omega} = \omega_r + i\omega_i$ are complex.

ω_r → real part, ω_i → imaginary part

So $$\frac{dx}{dt} = \text{Re}\left\{i\underline{\omega}\,\underline{A}\, e^{i\underline{\omega}t}\right\}$$

$$\frac{d^2x}{dt^2} = \text{Re}\left\{-\underline{\omega}^2\underline{A}\, e^{i\underline{\omega}t}\right\}$$

Plugging these values for x and its derivatives in eqn. (1) gives us :

$$\mathrm{Re}\left\{ -m\underline{\omega}^2 A\, e^{-\omega_i t} e^{i(\omega_r t+\phi_0)} + i(\omega_r + i\omega_i)\, bmA\, e^{-\omega_i t} e^{i(\omega_r t+\phi_0)} + KA\, e^{-\omega_i t} e^{i(\omega_r t+\phi_0)} \right\} = 0$$

or, $$\mathrm{Re}\left\{ -m(\omega_r^2 - \omega_i^2 + 2i\omega_r\omega_i) A e^{-\omega_i t} e^{i(\omega_r t+\phi_0)} + i\omega_r bmA e^{-\omega_i t} e^{i(\omega_r t+\phi_0)} - \omega_i bmA e^{-\omega_i t} e^{i(\omega_r t+\phi_0)} + KAe^{-\omega_i t} \times e^{i(\omega_r t+\phi_0)} \right\} = 0$$

Separating the real part and equating it to zero, we get :

$$\left\{ \omega_i^2 - \omega_r^2 - \omega_i b + \frac{K}{m} \right\} \cos(\omega_r t + \phi_0) + \left\{ 2\omega_r\omega_i - \omega_r b \right\} \sin(\omega_r t + \phi_0) = 0$$

The only way the above expression can vanish is if:

$$\begin{cases} \omega_i^2 - \omega_r^2 - \omega_i b + \frac{K}{m} = 0 \\ \& \quad 2\omega_r\omega_i - \omega_r b = 0 \end{cases}$$

which has the solution :

$$\omega_r = \pm\sqrt{\frac{K}{m} - \frac{b^2}{4}}$$

$$\omega_i = \frac{b}{2}$$

So the general solution is :

$$x = \mathrm{Re}\left\{ x_{eq} + \underline{A}\; e^{-bt/2}\, e^{\pm i\sqrt{\frac{K}{m} - b^2/4}} \right\}$$

or, $$\boxed{x = x_{eq} + A e^{-bt/2} \cos\left[\left(\sqrt{\frac{K}{m} - b^2/4} \right) t + \phi_0 \right]}$$

There are 2 adjustable parameters : A & ϕ_0 .

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(c) $b = 0.5\ s^{-1}$, $K = 632\ N/m$, $m = 1\,kg$, $x_{eq} = 0$

So $\sqrt{\frac{K}{m} - \frac{b^2}{4}} \simeq 25.14\ s^{-1}$

$$\therefore \quad x = A\, e^{-t/4\,sec} \cdot \cos\left(25.14\ sec^{-1} \times t + \phi_0\right)$$

Next use the x_0 & v_0 information.

$$x_0 = A \cos\phi_0 = 5\ mm \qquad \text{—— (2)}$$

$$v_0 = \left.\frac{dx}{dt}\right|_{t=0}$$

$$= A\left\{ -\frac{b}{2} e^{-bt/2} \cos\left(\sqrt{\tfrac{K}{m} - \tfrac{b^2}{4}}\ \cdot t + \phi_0\right) - e^{-bt/2} \times \sqrt{\tfrac{K}{m} - \tfrac{b^2}{4}}\ \sin\left(\sqrt{\tfrac{K}{m} - \tfrac{b^2}{4}}\ \cdot t + \phi_0\right) \right\} \Bigg|_{t=0}$$

$$= A\left(-\frac{1}{4}\cos\phi_0 - 25.14 \sin\phi_0\right)$$

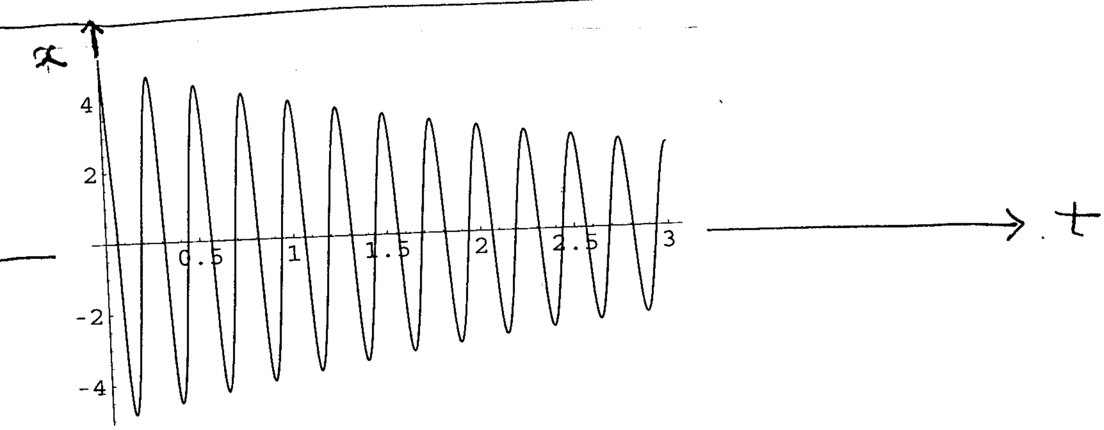
But $v_0 = 0$

$$\Rightarrow \quad \frac{A}{4}\cos\phi_0 = -A \times 25.14 \sin\phi_0 \qquad \text{—— (3)}$$

From (2) & (3), we get :-

$$\boxed{\begin{cases} \phi_0 = -0.01\ \text{radians} \\ A \simeq 5\ mm \end{cases}}$$

$$\therefore \quad x = 5\, e^{-t/4\,sec} \cos\left(25.14\ sec^{-1} \times t - 0.01\right)$$



3 Shake it up!

(a) Mass of radio reciever $= m = 5.24$ Kg

Frequency of vibrations $= f = 120$ Hz

Maximum acceleration $= a_{max} = 98\ m/s^2$

Assume a solution of the form:

$$x(t) = x_{eq} + A \cos(\omega_0 t + \phi_0) \quad \text{——— (4)}$$

Then maximum speed $= \omega_0 A = 2\pi f A$

maximum acceleration $= \omega_0^2 A = 4\pi f^2 A$

(This follows simply from the expressions for $\frac{dx}{dt}$ & $\frac{d^2x}{dt^2}$).

$$a_{max} = 98\ m/s^2$$

$$\Rightarrow \quad A = \frac{a_{max}}{4\pi f^2} = \underline{\underline{0.17\ mm}}$$

$$v_{max} = \omega_0 A = \underline{\underline{0.13\ m/s}}$$

(b) At $t=0$, $x = x_{eq} + 0.12$ mm moving along $-x$ direction

$$\Rightarrow \quad A \cos\phi_0 = 0.12\ mm \quad (\text{from eqn } (4))$$

$$\Rightarrow \quad \cos\phi_0 = \frac{0.12\ mm}{0.17\ mm} = 0.706$$

$$\& \quad \sin\phi_0 = \sqrt{1-\cos^2\phi_0} = 0.708$$

$$\therefore \text{ Complex amplitude} = \underline{A} = A_{real} + i\, A_{imaginary}$$

$$= A(\cos\phi_0 + i \sin\phi_0)$$

$$\simeq \underline{\underline{0.12\ mm + i\ 0.12\ mm}}$$

4 An unusual drag force :

(a) Equation of motion :

$$\boxed{m\frac{d^2x}{dt^2} + K(x - x_{eq}) + C\frac{d^3x}{dt^3} = F_0 \cos\omega t}$$

(b) Let $x = x_{eq} + \mathrm{Re}\{\underline{A}\, e^{i\omega t}\}$.

So
$$\frac{dx}{dt} = \mathrm{Re}\{i\omega \underline{A}\, e^{i\omega t}\}$$
$$\frac{d^2x}{dt^2} = \mathrm{Re}\{-\omega^2 \underline{A}\, e^{i\omega t}\}$$
$$\frac{d^3x}{dt^3} = \mathrm{Re}\{-i\omega^3 \underline{A}\, e^{i\omega t}\}$$

So we can write the equation of motion as :-

$$\mathrm{Re}\left[\left(-i\omega^3 C\underline{A} - \omega^2 m\underline{A} + K\underline{A} - F_0\right) e^{i\omega t}\right] = 0$$

This must hold true for all times .

So $$-i\omega^3 C\underline{A} - \omega^2 m\underline{A} + K\underline{A} - F_0 = 0$$

$$\Rightarrow \quad \underline{A} = \frac{F_0}{\left[K - m\omega^2 \cancel{+} - iC\omega^3\right]}$$

$$\Rightarrow \quad A = \frac{F_0}{\sqrt{(K - m\omega^2)^2 + C^2\omega^6}}$$