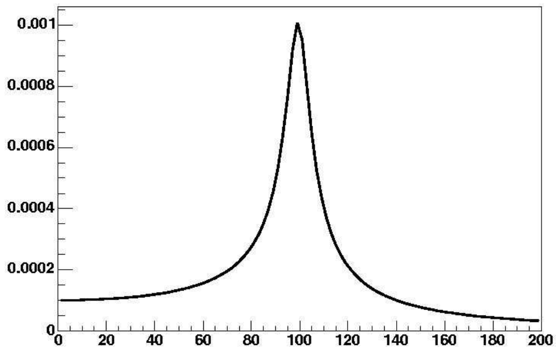
1 Resonance for an unusual oscillator

$$\underline{A} = \frac{F_0}{k - m\omega^2 - iC\omega^3} = \frac{F_0}{m(\omega_0^2 - \omega^2) - iC\omega^3}$$

(a) The amplitude of the oscillator, as a function of w, is the magnitude of the complex number $\underline{A}$:

$$A(\omega) = |\underline{A}(\omega)| = \left| \frac{F_0}{m(\omega_0^2 - \omega^2) - iC\omega^3} \right| = \frac{|F_0|}{\sqrt{(m(w_0^2 - \omega^2))^2 + (C\omega^3)^2}}$$

(b) Yes! The resonance is the peak around $\omega_0 = 100s^{-1}$:



$A(\omega)$ in meters vs. ω in inverse seconds

2 Seismograph

a) The equation of motion is

$$\begin{aligned} ma &= \sum F \\ &= \text{spring force} + \text{damping force} \end{aligned}$$

Since "the spring force and the damping force depend on the displacement and velocity relative to the earth's surface," the equation becomes

$$ma = -ky - \gamma m \frac{dy}{dt}.$$

If you are wondering what happened to the gravitational force, it is actually included in $-ky$. The constant force due to gravity just shifts the equilibrium

point down a distance mg/k. In writing $-ky$ rather than $-k(y - y_{eq})$, we have chosen the origin to be at this equilibrium point.
Now, the "dynamically significant acceleration is the acceleration of M relative to the fixed stars." This means that

$$a = \frac{d^2}{dt^2}(y + \eta).$$

Now substitute for a.

$$m\frac{d^2}{dt^2}(y + \eta) = -ky - \gamma m\frac{dy}{dt}$$

Finally, introduce $\omega_0^2 = k/m$ and rearrange the terms.

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + w_0^2 y = -\frac{d^2\eta}{dt^2}$$

b) Let $\eta = C\cos(wt)$ to get

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = -\omega^2 C\cos(\omega t).$$

Now generalize to a complex differential equation ($\cos(\omega t) \to e^{i\omega t}$).

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = -\omega^2 C e^{i\omega t}.$$

Guess a solution of the form $y = \underline{A}e^{iwt}$. Substituting for y transforms the differential equation into an algebraic equation.

$$(-\omega^2 + \gamma i\omega + \omega_0^2)\underline{A}e^{i\omega t} = -\omega^2 C e^{i\omega t}$$

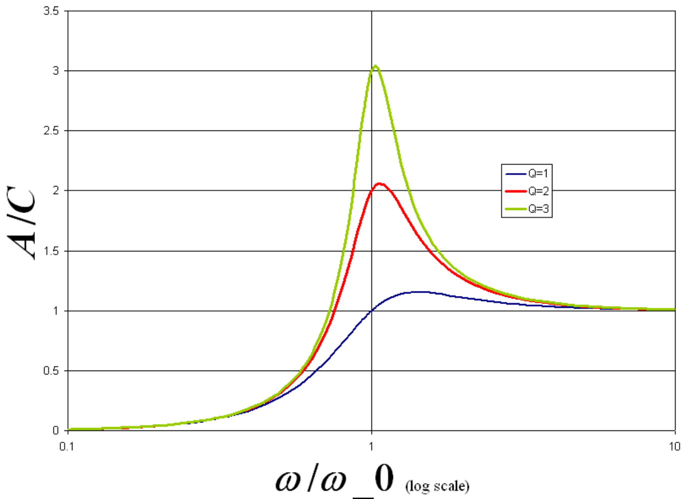
After canceling the $e^{i\omega t}$, the equation may be solved for $\underline{A}$.

$$\underline{A} = \frac{-Cw^2}{-\omega^2 + \gamma i\omega + \omega_0^2}$$

The trial solution $y = \underline{A}e^{i\omega t}$ will work. The solution to the original real differential equation is the real part of y:

$$\text{physically significant solution} = \Re\, y = \Re\left(\frac{-Cw^2}{-\omega^2 + \gamma i\omega + \omega_0^2}e^{i\omega t}\right)$$

c) For a graph of $|\underline{A}| = \frac{|w^2 C|}{\sqrt{(w_0^2 - w^2)^2 + (\gamma\omega)^2}}$, we need to choose values for C, ω_0, and γ. However, it is more instructive to plot the normalized quantities A/C and ω/ω_0. Then all we have to choose is the value of $\gamma/\omega_0 \equiv 1/Q$. From part (d), a realistic value of Q is 2. Below, we show the normalized plots for $Q = 1$, 2, and 3.



A normalized plot of $|A(\omega)|/C$ vs. ω/ω_0 for Q=1, 2, 3.

d) Suppose the natural period of the seismograph with $Q = \omega_0/\gamma = 2$ is $T_0 = 30s$. Suppose the earth shakes with a period $T = 20min = 20 \times 60s = 1200s$ and with a maximum acceleration of $10^{-9}m/s^2$. The question is asking for the amplitude of the oscillator.

First find the angular frequency of the earth's vibration

$$T = 1200s \rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{1200s} = 0.005s^{-1}$$

The natural frequency is

$$w_0 = \frac{2\pi}{T_0} = \frac{2\pi}{30s} = 0.209s^{-1}.$$

and the damping constant is

$$\gamma = \frac{w_0}{Q} = \frac{0.209s^{-1}}{2} = 0.105s^{-1}.$$

The amplitude is then

$$\begin{aligned}|\underline{A}| &= \frac{|Cw^2|}{\sqrt{(w_0^2 - w^2)^2 + (\gamma\omega)^2}} \\ &= \frac{10^{-9}m/s^2}{\sqrt{((0.209s^{-1})^2 - (0.005s^{-1})^2)^2 + (0.105s^{-1} \times 0.005s^{-1})^2}} \\ &= 2.29 \times 10^{-8}m.\end{aligned}$$

3 Energy in a driven oscillator

$$x(t) = A\cos(\omega t + \phi_0)$$
$$A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + (b\omega)^2}}$$
$$\phi_0 = -\arctan\frac{b\omega}{\omega_0^2 - \omega^2}$$
$$f = -bmv$$
$$F(t) = F_0\cos(\omega t)$$

For reference:

$$v(t) = \frac{dx}{dt}(t) = -\omega A\sin(\omega t + \phi_0)$$

(a) Power dissipated by the damping force

$$P_f = f \cdot v = -bmv \cdot v = -bm\left(-\omega A\sin(\omega t + \phi_0)\right)^2 = -bm\omega^2 A^2 \sin^2(\omega t + \phi_0)$$

(b) Since $< \sin^2 >= \frac{1}{2}$

$$< P_f >= -\frac{1}{2}bm\omega^2 A^2$$

(c) P_F

$$P_F = F \cdot v = F_0\cos(\omega t) \cdot -\omega A\sin(\omega t + \phi_0) = -F_0 A\omega\cos(\omega t)\sin(\omega t + \phi_0)$$

(d)

$$\begin{aligned} P_F &= -F_0 A\omega\cos(\omega t)\left(\sin(\omega t)\cos(\phi_0) + \cos(\omega t)\sin(\phi_0)\right) \\ &= -F_0 A\omega\cos(\omega t)\sin(\omega t)\cos(\phi_0) - F_0 A\omega\cos(\omega t)\cos(\omega t)\sin(\phi_0) \end{aligned}$$

After noting that $\cos(\omega t)\sin(\omega t) = \frac{1}{2}\sin(2\omega t)$ averages to zero, the average value is:

$$< P_F >= -\frac{1}{2}F_0 A\omega\sin(\phi_0)$$

(e) Conservation of energy would suggest that

$$< P_f > + < P_F >= 0$$

or

$$-\frac{1}{2}bm\omega^2A^2 - \frac{1}{2}F_0A\omega\sin(\phi_0) = 0$$

The trick is to realize that

$$\sin(\phi_0) = \frac{-b\omega}{\sqrt{(\omega^2-\omega_0^2)^2+(b\omega)^2}}.$$

This comes from $\phi_0 = -\arctan\frac{b\omega}{\omega_0^2-\omega^2}$. Substituting gives:

$$-\frac{1}{2}bm\omega^2A^2 - \frac{1}{2}F_0A\omega\left(\frac{-b\omega}{\sqrt{(\omega^2-\omega_0^2)^2+(b\omega)^2}}\right) = 0$$

or

$$-\frac{1}{2}bm\omega^2A^2 - \frac{1}{2}F_0A\omega\left(-\frac{bAm\omega}{F_0}\right) = 0$$

4 Conversion of wave quantities

(a) Given: $v = 2.5m/s$ and $\lambda = 7.5m$
$\lambda f = v \rightarrow f = \frac{v}{\lambda} = \frac{2.5m/s}{7.5m} = 0.333s^{-1}$
$\kappa = 1/\lambda = 1/7.5m = 0.133m^{-1}$
$k = 2\pi\kappa = 2\pi \times 0.133m^{-1} = 0.838m^{-1}$
$\omega = 2\pi f = 2\pi \times 0.333s^{-1} = 2.094s^{-1}$

b) We need to convert frequency (f) to wavelength ($\lambda = v/f$). The conversion factor is the speed of sound at 20 degrees Celsius ($v = 343.6m/s$)[1].

$$\lambda(20\text{Hz}) = \frac{343.6m/s}{20Hz} = 17.18m$$

$$\lambda(20,000\text{Hz}) = \frac{343.6m/s}{20,000Hz} = 1.718cm$$

5 Plucking a harpsichord string

$$x_{left} = 20cm$$

$$x_{right} = 160cm$$

$$x_{left} + x_{right} = 180cm$$

$$y = 0.4cm$$

(a) There are (at least) two ways to interpret this question that give very nearly the same answers. One is to assume that when the plectrum lifts the

[1]see `http://hyperphysics.phy-astr.gsu.edu/hbase/sound/souspe.html`

string, the string slips over the plectrum, allowing the hypotenuse to increase a bit. In this case, the plectrum moves straight up and the slopes are

$$s_{left} = \frac{y}{x_{left}} = 0.02000$$
$$s_{right} = -\frac{y}{x_{right}} = -0.0025$$

Another way to interpret the problem is that the string doesn't slip over the plectrum. Thus, the plectrum shifts to the left ever so slightly along with the chunk of string that it lifts. In this situation the exact slope on left is

$$s_{left} = \tan(\arcsin(\frac{y}{x_{left}})) = 0.02000.$$

The difference between the two is very small! The new calculation of the slope on the right uses the result from part b).

$$s_{right} = -\frac{y}{\Delta x + x_{right}} = -\frac{.4cm}{0.004cm + 180cm} = -0.0022$$

and again is close to the first method. [Sorry for invoking the result of part b, we did not anticipate this type of solution for part (a). The point here is that both methods agree because y is very small.]

(b) The change in position is (assuming the first situation where the string slips a bit)

$$\begin{aligned}
|\Delta x| &= x_{initial} - x_{final} \\
&= x_{left} - x_{left}\frac{x_{left}}{\sqrt{y^2 + x_{left}^2}} \\
&= x_{left}\left(1 - \frac{x_{left}}{\sqrt{y^2 + x_{left}^2}}\right) \\
&= 20cm\left(1 - \frac{20cm}{\sqrt{(0.4cm)^2 + (20cm)^2}}\right) \\
&= 20cm\,(1 - 0.9998) \\
&= 0.004cm
\end{aligned}$$

The second method where x_{left} becomes the hypotenuse is

$$\begin{aligned}
|\Delta x| &= x_{initial} - x_{final} \\
&= x_{left} - \sqrt{x_{left}^2 - y^2} \\
&= 20cm - \sqrt{(20cm)^2 - (0.4cm)^2} \\
&= 0.004cm
\end{aligned}$$

(c) In the small-amplitude approximation, we assume the amplitude of the wave is small enough that motion of chunks in the x direction is negligible compared to the motion in the y direction. Part (b) showed this for one chunk of string. An equivalent way to look at it is to examine the slope of the string, as in (a). As long as the slope everywhere is $\ll 1$, the amount of string pulled through the fictitious hole at the right end is small and, again, x motion is negligible. (In the more realistic situation where the string is fixed at both ends, the small-amplitude approximation also lets us assume that the tension in the string is constant.)