Cornell University

Department of Physics

Phys 214 September 16, 2004

Waves, Optics, and Particles, Fall 2004

Homework Assignment # 4

(Due Thursday, September 23 at 5:00pm *sharp.*)

Agenda and readings for the week of September 20:

Skills to be mastered:

- distinguishing between speed of wave propagation and particle speed;
- understanding that the longitudinal component of tension τ_x is constant;
- getting τ_y knowing $\frac{\partial y}{\partial x}$;
- understanding that the "chunk" mass is $\mu\Delta x$;
- converting expressions like $\frac{\frac{\partial y}{\partial x}(x+\Delta x,t)-\frac{\partial y}{\partial x}(x,t)}{\Delta x}$ to the proper partial derivatives, $\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}(x,t)\right)$, in the limit $\Delta x \to 0$;
- interpretting "snapshots" $y(x, t = t_0)$ and "particle histories" $y(x = x_0, t)$;
- understanding the boundary conditions for both fixed and free ends of a string;
- sketching allowed modes for different types of boundary conditions;
- reading the wavelength off of pictures of different standing wave modes;
- extracting allowed frequencies from allowed wavelengths;
- confirming a given $y(x, t)$ as a solution to the wave equation;
- deriving the dispersion relation $\omega = f(k)$ from the standing wave form $y(x, t) = A\cos(\omega t)\cos(kx)$;
- relating the wave speed (v) to the physical parameters of the system in which the waves are propagating: i.e., $v = c \equiv \sqrt{\tau/\mu}$ for a string; $v = c \equiv \sqrt{B/\rho}$ for sound.

Lectures and Readings:

- Lec 8, 09/21 (Tue): Maxwell's equations in vacuum; Polarization Rule I for electromagnetic waves; differential forms for Maxwell's equations.
  Readings: LN "Other waves" Sec. 3–3.2.4; AG 27.1–27.5 (or YF 32.1–32.3.
- Lec 9, 09/23 (Thu): "Let there be light"; Polarization Rule II for light.
  Readings: LN "Other waves" Sec. 3.2.5-4; .

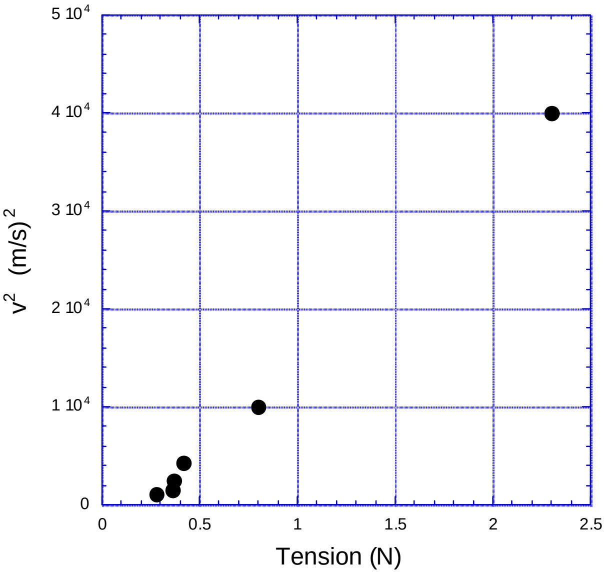

Figure 1: Wavespeed squared vs. tension for a harpsichord string

1 Studying harpsichord strings

A harpsichord builder studies the characteristics of strings. Using a scale calibrated in newtons, he determines the tension (τ) needed to produce a given wavelength (λ) in the vibration. For each result he computes the wave speed, using the fact that his apparatus drives the string at $f = 120.0$ Hz, and then makes a graph of v^2 vs. τ (Fig. 1).

(a) The builder neglected to zero the scale, so its readings are each high by a constant amount. How large is this offset error?

(b) What is the linear mass density (mass per unit length) of the string?

2 Chunk speed versus wave speed

A transverse wave on a string is described by $y(x,t) = (5.0 \text{ mm}) \cos(4.0\pi t - 1.0\pi x)$, where x is in meters and t is in seconds.

(a) What are the frequency and wavelength of the wave?

(b) What is the speed of propagation of the wave?

(c) What is the maximum "chunk speed" ($\partial y/\partial t$)?

(d) How do your answers to (b) and (c) compare? Do you think this relationship is generally true, as long as the amplitude of the wave is small? Explain.

(e) Find the maximum chunk acceleration.

3 Verifying the wave equation for a standing wave

A string under tension τ is fixed at both ends ($x = 0$ and $x = L$). A standing wave on the string is described by

$$y(x,t) = A\sin(kx)\cos(\omega t)\ . \tag{1}$$

Express all answers below in terms of the quantities τ, L, A, k, and ω. Don't leave μ or c in your answers.

(a) What are the possible values of k, given that the right end ($x = L$) is fixed?

(b) What is the y-component of the force due to the string on the fixed point $x = L$ at any time t?

(c) Consider a tiny *chunk* of string of length dx between x and $x + dx$. Find the x- and y-components of the force on the left side of this chunk (at x) due to the rest of the string.

(d) Find the x- and y-components of the force on the right side of this chunk (at $x + dx$) due to the rest of the string.

(e) Using your answers from (d) and (e), find the net force on the chunk.

(f) Verify that $\sum \vec{F} = m\vec{a}$ works for the chunk in the *limit* $dx \to 0$. (Note that you should be able to do better than saying $0 = 0$.)

4 Wave equation with gravity

Take the same string system from class lecture and the lecture notes, but do not ignore the gravitational force acting on each chunk of the string.

(a) Redraw the force-body diagram of Figure 2 from the lecture notes "Intro to Waves," this time including the gravitational force. Explain (briefly, one or two short sentences) why this force does not affect motion in the x direction and hence does not modify Eq. (7) in the notes.

(b) Following the analysis of Section 4.2.2 of the lecture notes, rewrite Eq. (8), Eq. (11), and Eq. (13) to take into account the presence of gravity. Are any of these equations unchanged by the presence of gravity?

(c) The string sags a *tiny* bit due to gravity. Use the equation of motion to find the function $y_0(x)$ that gives the shape of the string when there is no wave present. (Hint: what is the chunk acceleration in the absence of any wave on the string?) The string has fixed ends $y(x = 0) = y(x = L) = 0$.

(d) We might guess small amplitude standing waves to have the form

$$y(x,t) = y_0(x) + A\sin kx\cos\omega t \tag{2}$$

where $y_0(x)$ is the equilibrium position of each chunk as found in part (c). Is this guess a solution to the equation of motion?

5 Normal modes in a thin tube

A 2.0-m-long thin tube is open at both ends. With a loudspeaker and frequency generator, you measure the lowest resonant frequency to be 85 Hz.

(a) What are the next two lowest resonant frequencies?

(b) A small microphone that responds to pressure is used to examine the standing waves in the tube. At what values of x does the microphone output show a maximum amplitude for the 5th lowest resonant frequency?

(c) The frequency generator is tuned to the 3rd lowest resonant frequency. Then without changing the frequency, an additional length of tube is connected to one end. What is the shortest length of tube added that will restore resonance?

(d) The original tube (2.0 m long) is closed at one end. What are the three lowest resonant frequencies now?

6 Inharmonicity of piano strings

Most piano "strings" are steel wires. The stiffness of the steel—that is, its resistance to bending—has to be taken into account to accurately describe transverse waves on a piano string. The propagation of transverse waves along such strings is described by the modified wave equation

$$\mu \frac{\partial^2 y}{\partial t^2} - \tau \frac{\partial^2 y}{\partial x^2} + \Gamma \frac{\partial^4 y}{\partial x^4} = 0 \tag{3}$$

where μ and τ (as in lecture) are the the linear mass density and tension, respectively. Γ is an elastic parameter that depends on the diameter of the wire and the stiffness of the material, but is independent of the tension.

(a) Show that the standing wave $y(x,t) = A\cos(\omega t)\cos(kx)$ is a solution to the modified wave equation (3) and derive the dispersion relation $\omega = f(k)$.

(b) What are the allowed values of k_n in terms of the length of the string L?

(c) In a piano string without stiffness, the frequencies of the standing wave modes would be integral multiples of the fundamental ($\omega_n = n\omega_1$): in other words, they would form a "harmonic series". Stiffness in the wire causes the frequencies to deviate from a harmonic series, a phenomenon called *inharmonicity*. Inharmonicity is an important element in the tone quality of a piano. From the dispersion relation and the allowed values of k_n, find the relationship between ω_n and ω_1 for a piano string with stiffness. [Note that, in the relationship between ω_n and ω_1, the value of ω_1 includes the effect of stiffness on the fundamental frequency. The expression we want you to derive is for the deviation of the higher frequency modes from ideal harmonic behavior, given a fundamental frequency ω_1. So your answer should take the form $\omega_n = Cn\omega_1$, where C is a "correction factor" that is equal to 1 for $n = 1$.]

(d) If the elastic parameter Γ is sufficiently small, show that an approximate relation between ω_n and ω_1 is

$$\omega_n = n\omega_1 \left(1 + (n^2 - 1)\frac{\Gamma\pi^2}{2\tau L^2}\right) \tag{4}$$

[Hint: use the binomial approximation: if $\epsilon \ll 1$, $(1+\epsilon)^n \approx 1 + n\epsilon$.]

(e) Explain in physical terms why stiffness has more effect for higher n. In other words, why does the "correction factor" $1 + (n^2 - 1)\frac{\Gamma\pi^2}{2\tau L^2}$ increase with n?