Physics 214 PS 4 Solutions

1. Studying harpsichord strings

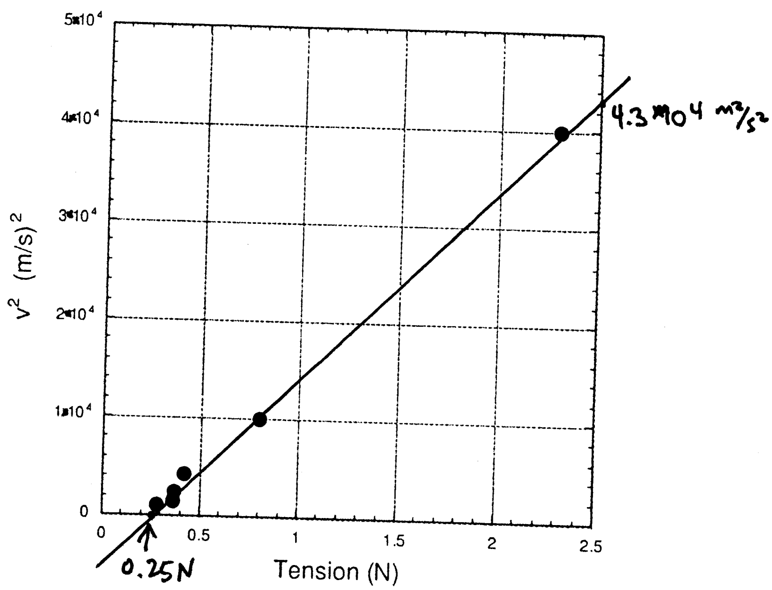


Figure 1: Wavespeed squared vs. tension for a harpsichord string

(a) We draw a fit line through the data, with equation

$$v^2 = m(T - T_0)$$

where m is the slope of the line, and T_0 is the tension when the $v^2 = 0$. [the left side would be $v^2 - v_0^2$ but we define $v_0^2 = 0$] We read $\boxed{T_0 = 0.25\,N}$ off the graph; this is his offset error.

(b) We calculate the slope,

$$m = \frac{\Delta v^2}{\Delta T} = \frac{(4.3\times10^4\ m^2/s^2 - 0)}{(2.5N - 0.25N)} = 1.9\times10^4\ \frac{m}{kg}.$$

Thus,

$$v^2 = \left(1.9\times10^4\ \frac{m}{kg}\right)T.$$

But we know that $v^2 = \frac{T}{\mu}$, so

$$\boxed{\mu = \frac{1}{1.9\times10^4\ m/kg} = 5.23\times10^{-5}\ kg/m = 52\ mg/m}$$

2. Chunk speed versus wave speed

$$y(x,t) = (5.0\,\mathrm{mm})\cos\left(\frac{4.0\pi}{s}t - \frac{1.0\pi}{m}x\right)$$

(a) We read off the frequency (angular) and wave vector from the equation, which is in standard form:

$$y(x,t) = A\cos(\omega t - kx),$$

so $\omega = 4.0\pi\ s^{-1}$ and $k = 1.0\pi\ m^{-1}$. Now we have

$$\boxed{\begin{aligned} f &= \frac{\omega}{2\pi} = 2.0\ \mathrm{Hz} \\ \lambda &= \frac{2\pi}{k} = 2.0\ \mathrm{m} \end{aligned}}$$

(b) The speed is defined by

$$\boxed{c = \frac{\omega}{k} = \frac{\lambda}{T} = f\lambda = 4.0\ \mathrm{m/s}},$$

where we use the last expression and plug in for f & λ.

(c) The "chunk speed" is defined as the speed of a single particle on the string. We know that the x-position of the particles don't change, so only the y-movement matters:

$$v_{chunk} = \frac{\partial y}{\partial t} = -(5.0\,\mathrm{mm})\left(\frac{4.0\pi}{s}\right)\sin\left(\frac{4.0\pi}{s}t - \frac{1.0\pi}{m}x\right).$$

The maximum v_{chunk} is simply the amplitude,

$$\boxed{(v_{chunk})_{max} = \left(\frac{\partial y}{\partial t}\right)_{max} = (5.0\,\mathrm{mm})\left(\frac{4.0\pi}{s}\right) = 20\pi\ \frac{\mathrm{mm}}{s} \approx 0.063\ \mathrm{m/s}}$$

(d) $\boxed{v_{chunk} \ll c}$. The wave travels significantly faster than the individual molecules. This is generally the case, since the chunk speed depends on (and decreases with) the amplitude, while the wave speed is constant, depending only on k and ω.

(e) We differentiate twice

$$\boxed{(a_{chunk})_{max} = \left(\frac{\partial^2 y}{\partial t^2}\right)_{max} = (5.0\,\mathrm{mm})\left(\frac{4.0\pi}{s}\right)^2 = 80\pi^2\ \frac{\mathrm{mm}}{s^2} \approx 0.79\ \mathrm{m/s^2}}$$

3. Verifying the wave equation for a standing wave

(a) The prescribed boundary conditions require

$$y(0,t) = y(L,t) = 0.$$

The wavefunction $y(x,t) = A\sin(kx)\cos(\omega t)$ satisfies this only for certain values of k, and must be valid at all times t. Thus,

$$\sin(kL) = 0$$

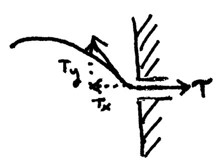
So that

$$kL = 0, \pi, 2\pi, \dots = n\pi \quad (n = 0, 1, 2, \dots).$$

Thus,

$$\boxed{k = \frac{n\pi}{L} \quad (n = 0, 1, 2, \dots)}$$

The $y(0,t) = 0$ boundary condition is satisfied "for free" by our choice of $\sin(kx)$ in the wavefunction.

(b) 

The x-component of tension is defined

$$T_x(L) = -\tau$$

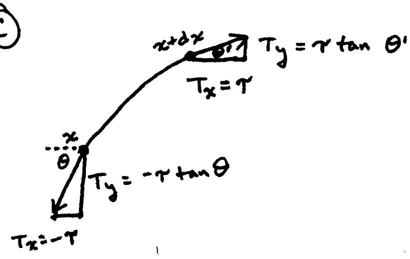
by our problem statement. The slope of the string relates $T_x(L)$ to $T_y(L)$:

$$\frac{T_y(L)}{T_x(L)} = (\text{slope}) = \frac{\partial y}{\partial x} = Ak\sin(kx)\cos(\omega t).$$

So $T_y(L) = T_x(L)\, Ak\cos(kL)\cos(\omega t)$.

If $\sin(kL) = 0$ then $\cos(kL) = \pm 1$. So we do not know the sign of the result. But we know

$$\boxed{|T_y(L)| = \tau Ak\,|\cos(\omega t)|}, \quad \text{or} \quad \boxed{T_y(L) = -\tau Ak\cos(kL)\cos(\omega t)}.$$

(c) 

On the left side (x), we find $\boxed{T_x(x) = -\tau}$, since the force must balance.

The vertical component, $T_y(x) = T_x(x)\tan\theta$ is just the slope times the x-component,

$$\boxed{T_y(x) = -\tau\frac{dy}{dx} = -\tau Ak\cos(kx)\cos(\omega t)}$$

(d) From the same picture and similar argument,

$$\boxed{T_x(x+dx) = \tau}$$

is to the right and

$$\boxed{T_y(x+dx) = \tau Ak\cos(k[x+dx])}$$

3 (cont'd).

(e) The net force is found by adding the forces acting on the left and right sides:

$$\boxed{(F_{net})_x = \tau - \tau = 0}$$

$$\boxed{(F_{net})_y = k\tau A\left(\cos(\omega t)\right)\left(\cos[k(x+dx)] - \cos[kx]\right)}$$

$$= k\tau A \cos(\omega t)\left(\cos(kx)\cdot\cos(kdx) - \sin(kx)\cdot\sin(kdx) - \cos(kx)\right)$$

$$\approx k\tau A\cos(\omega t)\left(-kdx\,\sin(kx)\right) = \boxed{-k^2\tau A\cos(\omega t)\sin(kx)\,dx \approx (F_{net})_y}$$

constitutive relation!

There were a number of ways to simplify $(\cos k(x+dx) - \cos(kx))$. We could have viewed it as a derivative, and multiplied by dx. Or we could expand it with angle addition rules, realizing that $\cos(kdx) \approx 1$ and $\sin(kdx) \approx kdx$. I chose the latter.

(f) As $(dx \to 0)$ we have a very small net force, but (ma) is also very small, since the mass is infinitessimal as well. Thus,

$$m = \mu dx,$$

$$\vec{a} = \frac{d^2y}{dt^2}\hat{y} = -\omega^2 A\sin(kx)\cos(\omega t)\hat{y},$$

so that

$$m\vec{a} = -\mu\omega^2 \underline{A\sin(kx)\cos(\omega t)\,dx\,\hat{y}}.$$

From above,

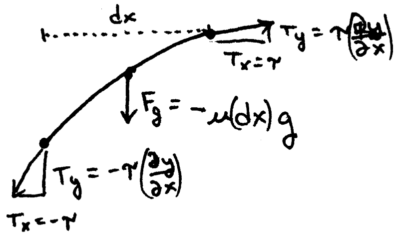
$$\vec{F}_{net} = \sum \vec{F} = -k^2\tau \underline{A\cos(\omega t)\sin(kx)\,dx\,\hat{y}}.$$

The underlined parts are the same in both. Thus they are equal if and only if

$$\mu\omega^2 \stackrel{?}{=} k^2\tau.$$

But $\omega^2 = c^2k^2$ and $\mu = c^2\tau$, so they are equal. Thus, as $dx \to 0$, Newton's second law is satisfied so the wave equation is indeed valid.

4. <u>Wave equation with gravity</u>

ⓐ 

We separate the forces into x- and y-components. The x-equation still cancels the same $(\tau - \tau = 0)$ since there are no new x-forces.

ⓑ Equation (8) is from $(F_{net})_y = \mu\, dx \frac{\partial^2 y}{\partial t^2}$. $(F_{net})_y$ now has an extra term, $-\mu\, dx\, g$, so after dividing out dx we get

$$\boxed{(8) \Rightarrow \frac{\partial T_y}{\partial x} - \mu g = \mu \frac{\partial^2 y}{\partial t^2}}.$$

Equation (11) is relating the T_x and T_y $\Rightarrow \frac{T_y}{T_x} = \frac{\partial y}{\partial x}$. This remains unchanged:

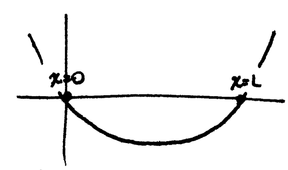
$$\boxed{(11) \Rightarrow T_y = \tau \frac{\partial y}{\partial x}}$$

Equation (13) is the result of combining (8) & (11):

$$\boxed{(13) \Rightarrow \frac{\partial^2 y}{\partial x^2} - \frac{\mu g}{\tau} = c^2 \frac{\partial^2 y}{\partial t^2}}$$

ⓒ If there is no wave, we get $y(x,t) = y_0(x)$. This means $\frac{\partial y_0}{\partial t} = 0$ so (13) becomes

$$\frac{\partial^2 y_0}{\partial x^2} = \frac{\mu g}{\tau} \; (= \text{constant}).$$



This is solved by a parabola. It must open upwards since μg is positive. The boundary conditions require roots at $x=0$ and $x=L$. Integrating gives

$$y_0(x) = \frac{1}{2}\frac{\mu g}{\tau} x^2 + c_1 x + c_2$$

but substituting in $y_0(0) = y_0(L) = 0$ gives us values, so that

$$\boxed{y_0(x) = \frac{1}{2}\frac{\mu g}{\tau}(x)(x-L)}.$$

ⓓ We plug in our guess $y(x) = \frac{1}{2}\frac{\mu g}{\tau}(x)(x-L) + A\sin(kx)\cos(\omega t)$ into (13)

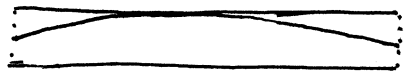
$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y_0}{\partial x^2} - Ak^2 \sin(kx)\cos(\omega t) = \frac{\mu g}{\tau} - Ac^2\omega^2 \sin(kx)\cos(\omega t)$$

$$c^2 \frac{\partial^2 y}{\partial t^2} = \left(c^2 \cancel{k} - A\omega^2 \sin(kx)\cos(\omega t)\right)$$

Clearly when we subtract $-\mu g$ from $\frac{\partial^2 y}{\partial x^2}$, it satisfies (13), the equation of motion.

5. Normal modes in a thin tube

(a) We have sound waves with open ends. Thus, the pressure must equal the outside pressure, so it has nodes at both ends.

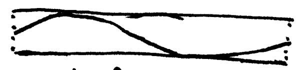


$f_0 = 85$ Hz.

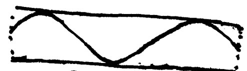
$L = 2.0$m $\Rightarrow \lambda = 4.0$m

* Note: these pictures may be deceptive — sound waves are not transverse, but we draw them as if they were. The y-axis here corresponds to pressure.

This corresponds to $L = \frac{\lambda}{2}$. The next two resonances are at $L = \lambda$ and $L = \frac{3\lambda}{2}$.



$L = \lambda = 2.0$m



$L = \frac{3\lambda}{2} \Rightarrow \lambda = \frac{2L}{3} = \frac{4}{3}\text{m} = 133$ cm

To find the frequencies we use

$$c = \lambda f = (4.0\text{ m})(85\text{ Hz}) = 340\text{ m/s}$$

to find the wave speed. Then $f = \frac{c}{\lambda}$ with the shorter wavelengths:

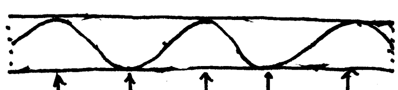
$$f_{1st} = f_0 = 85\text{ Hz}$$

$$\boxed{\begin{aligned} f_{2nd} &= \frac{340\text{ m/s}}{2.0\text{ m}} = 170\text{ Hz} = 2f_0 \\ f_{3rd} &= \frac{340\text{ m/s}}{133\text{ cm}} = 255\text{ Hz} = 3f_0 \end{aligned}}$$

(b) For the 5th lowest resonance, we have $f_5 = 5f_0 = 425$ Hz and $\lambda = \frac{2L}{5} = 80$ cm. We have maximal pressure at the antinodes,

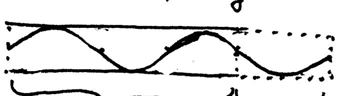
$\frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \frac{7\lambda}{4}, \frac{9\lambda}{4}$

= 20 cm, 60 cm, 100 cm, 140 cm, and 180 cm



$\boxed{\text{20cm} \quad \text{60cm} \quad \text{100cm} \quad \text{140cm} \quad \text{180cm}}$

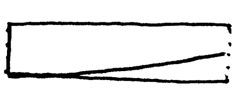
(c) The 3rd frequency is $f_3 = 255$ Hz, $\lambda = 133$ cm.



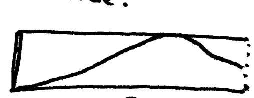
2m $\quad \lambda/2 = 66$cm.

We must add $\boxed{66\text{ cm} = \lambda/2}$ to the tube for another resonance.

(d) If we close one end then we have an antinode, since the displacement is a node.



$L = \lambda/4$



$L = 3\lambda/4$

$L = 5\lambda/4$

So $\lambda = 8.0$m, 2.66 m, 1.60m. The speed ($c = 340$ m/s) is the same, so $f = c/\lambda$ gives

$$\boxed{f_1 = 42.5\text{ Hz} \qquad f_2 = 127.5\text{ Hz} \qquad f_3 = 212.5\text{ Hz}}$$

6. Inharmonicity of piano strings

(a) We substitute the proposed solution,

$$y(x,t) = A\cos(\omega t)\sin(kx)$$

into the wave equation,

$$\mu \frac{\partial^2 y}{\partial t^2} - \tau \frac{\partial^2 y}{\partial x^2} + \Gamma \frac{\partial^4 y}{\partial x^4} = 0$$

and find

$$\mu\left(-A\omega^2 \cos(\omega t)\sin(kx)\right) - \tau\left(-Ak^2\cos(\omega t)\sin(kx)\right) + \Gamma\left(Ak^4\cos(\omega t)\sin(kx)\right) = 0.$$

Dividing out $-y$ yields

$$\mu\omega^2 - \tau k^2 - \Gamma k^4 = 0.$$

We solve for $\omega(k)$, using $c = \sqrt{\frac{T}{\mu}}$ as the of the unperturbed wave.

$$\boxed{\omega(k) = \sqrt{\frac{T}{\mu}k^2 + \frac{\Gamma}{\mu}k^4} = ck\sqrt{1 + \frac{\Gamma}{T}k^2}}.$$

This is the dispersion relation.

(b) If the string has length L, then $k_n = \frac{n\pi}{L}$, $n = 0, 1, 2, \ldots$ (see 3a).

(c) Substituting $\omega_n = \omega(k_n) = \frac{n\pi c}{L}\sqrt{1 + \frac{\Gamma}{\tau}\frac{n^2\pi^2}{L^2}}$. We have

$$\omega_1 = \frac{\pi c}{L}\sqrt{1 + \frac{\Gamma}{T}\frac{\pi^2}{L^2}}$$

so that

$$\boxed{\omega_n = n\omega_1\sqrt{\frac{1 + \frac{\Gamma n^2\pi^2}{\tau L^2}}{1 + \frac{\Gamma\pi^2}{\tau L^2}}}} = n\omega_1\sqrt{\frac{\tau L^2 + \Gamma n^2\pi^2}{\tau L^2 + \Gamma\pi^2}}$$

(d) We use binomial expansion twice. Note that

$$\left(1 + \frac{\Gamma\pi^2}{\tau L^2}\right)^{-1} \approx 1 - \frac{\Gamma\pi^2}{\tau L^2}$$

so that

$$\frac{1 + \frac{\Gamma n^2\pi^2}{\tau L^2}}{1 + \frac{\Gamma\pi^2}{\tau L^2}} \approx \left(1 + \frac{\Gamma n^2\pi^2}{\tau L^2}\right)\left(1 - \frac{\Gamma\pi^2}{\tau L^2}\right) \approx 1 + (n^2 - 1)\frac{\Gamma\pi^2}{\tau L^2}.$$

(ignoring the much smaller term of order Γ^2)

When we take the square root,

$$\sqrt{1 + (n^2-1)\frac{\Gamma\pi^2}{\tau L^2}} \approx 1 + (n^2-1)\frac{\Gamma\pi^2}{2\tau L^2}.$$

Thus,

$$\boxed{\omega_n = n\omega_1\left(1 + (n^2-1)\frac{\Gamma\pi^2}{2\tau L^2}\right)}.$$

(e) The correction factor increases for higher n because there is more bending – compare $n \gg 10$  with $n = 1$ . Thus the perturbation is a much stronger effect.