Cornell University

Department of Physics

Phys 214 September 23, 2004

Waves, Optics, and Particles, Fall 2004

Homework Assignment # 5

(Due Thursday, September 30 at 5:00pm *sharp*.)

Agenda and readings for the week of September 27:

Skills to be mastered:

- given the wave solution $y(x,t)$, be able to get and interpret "snapshots" $y(x,t=t_0)$ and "particle histories" $y(x=x_0,t)$;
- relating the wave speed c to the physical parameters of the system in which the waves are propagating: i.e., $c \equiv \sqrt{\tau/\mu}$ for a string; $c \equiv \sqrt{B/\rho}$ for sound.
- deriving boundary conditions for the ends of a string;
- sketching the allowed modes for different types of boundary conditions;
- picking the wavelength off of pictures of different standing wave modes;
- computing allowed frequencies from the allowed wavelengths.

Lectures and Readings:

- Lectures 10 and 11, 09/28 (Tue) and 09/30 (Thu): (IV. General Wave Phenomena) "Pulse equation" and its general solution.
  Readings: LN "Wave Phenomena I," Sec. 1–3; VW pp. 201–209 and 223–228; AG sections 25.3–25.4.

1 Massive ring boundary condition and normal modes

A string of length L is under tension τ and has linear mass density μ. The string is fixed at one end ($x=0$) while the other end ($x=L$) is attached to a ring of mass M that can slide up and down a rod without friction. (See Figure 1.)

(This problem might serve as a simplified model of a string in a piano, guitar, or violin. One end of the string is fixed while the other passes over a bridge. The function of the bridge is to transmit vibrations from the string to other parts of the instrument such as the sounboard of the piano or the top plate of the violin. The mass of our slip ring would represent the effective inertia of the bridge as seen by the string.)

(a) Use $\sum \vec{F} = m\vec{a}$ to find the equation of motion for the ring at $x=L$. Neglect gravity. (This equation is the mathematical statement of the boundary condition at $x=L$.)

(b) Consider the limit $M \to 0$. What does your answer to (a) tell you about the motion and/or the shape of the string at $x=L$ in this limit? What type of familiar boundary condition ("free" or "fixed") does this represent? In terms of the string length L, what are the allowed wave vectors k for the normal modes in this case?

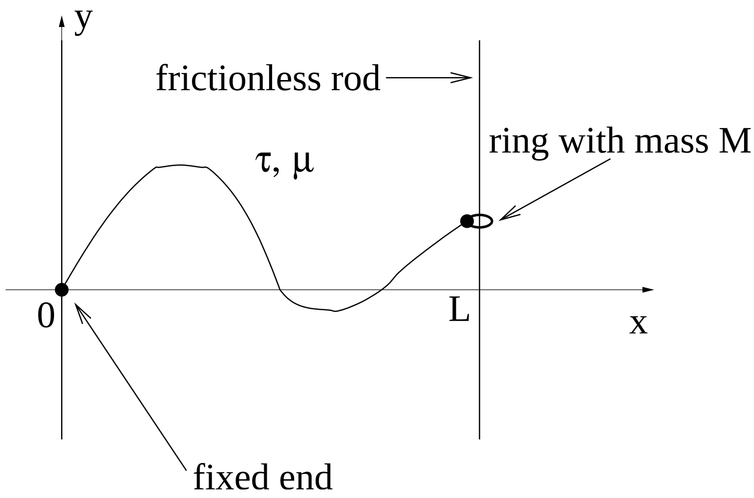

Figure 1: Boundary condition due to a ring of mass M at $x = L$.

(c) Now, consider the limit $M \to \infty$. What does your answer to **(a)** tell you about the motion and/or the shape of the string at $x = L$ in this limit? What type of familiar boundary condition ("free" or "fixed") does this represent? In terms of the string length L, what are the allowed wave vectors k for the normal modes in this case?

(d) Consider a standing wave of the form

$$y(x,t) = A\sin(kx)\cos(\omega t).$$

$y(x,t)$ satisfies the fixes boundary condition at $x = 0$, because $\sin kx$ is zero at $x = 0$. What must be true so that $y(x,t)$ also satisfies the boundary condition at $x = L$ found in part (a)? Your answer should be an equation involving k as well as some or all of the physical constants that describe the system (τ, μ, M, and L).

(Note: Solving this equation for k would tell us the possible values of the wave vector for this system. But you don't have to try to solve the equation for a general case. We will just solve it in our two limiting cases for M in the next two parts of the problem.)

(e) Show that the equation in (d) gives the expected values for k in the limit $M \to 0$.

(f) Show that the equation in (d) gives the expected values for k in the limit $M \to \infty$.

2 Damped Boundary Conditions for a Sound Tube

A tube of length L is filled with air of bulk modulus B and mass density ρ_0 at atmospheric pressure P_0. One end of the tube ($x = L$) is closed while the other end ($x = 0$) is attached to a *massless* piston of area A that can slide along the tube. (See Figure 2.) The piston is lubricated with oil; the oil exerts a horizontal viscous drag force on the piston equal to $f_x = -\beta v_x$.

We denote the sound displacement inside the tube—that is, the displacement of a "chunk" of air from its equilibrium position—as $s(x,t)$ and the pressure inside the tube as $P(x,t)$.

(a) Air Pressure and Displacement of the Piston

Draw the free body diagram for the piston, indicating the directions and the magnitudes of all the horizontal forces, using no quantities other than $P(x = 0, t)$, $s(x = 0, t)$, β, v_x, A, ρ_0, L, and P_0.
Note: You may not need *all* of these quantities.

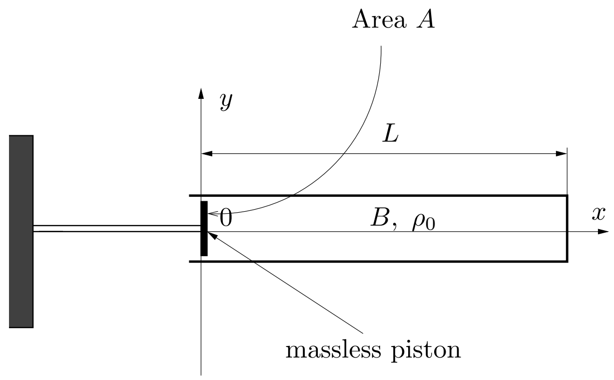

Figure 2: Generalized boundary conditions for a sound tube.

(b) Boundary Condition at the Piston

Starting with $\sum \vec{F} = m\vec{a}$, derive the equation of motion for the piston in terms of the degrees of freedom and known constants; i.e., using only

- the displacement function $s(x,t)$ and its derivatives evaluated at the $x = 0$ end of the tube; and
- some or all of β, A, B, ρ_0, L, and P_0.

(This equation is the mathematical statement of the boundary condition at $x = 0$.)

(c) Consider a general standing wave

$$s(x,t) = S_{\mathrm{m}} \sin(kx + \phi_0) \cos(\omega t).$$

S_{m} is the amplitude of the wave (i.e. the maximum displacement). Apart from the trivial solution in which $S_{\mathrm{m}} = 0$, can such a standing wave satisfy your boundary condition in (b) for a nonzero value of β? Explain your answer in terms of conservation of energy.

3 Prelim Review

There will be no problem set next week as we have a prelim coming. Try some of the problems from assignments and prelims given in previous semesters for extra practice and review. You might even use the first prelim given in the fall of 2001 as a practice prelim.