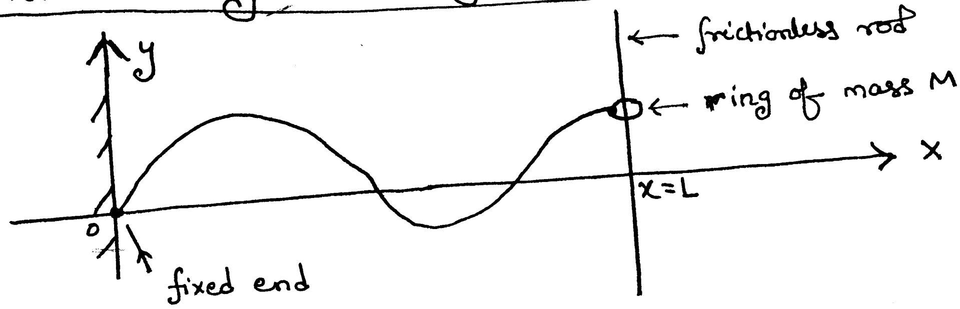
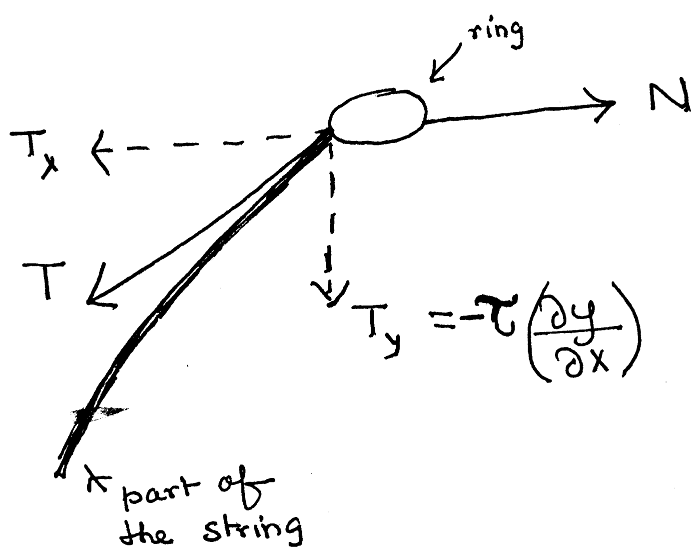
HOMEWORK SOLUTIONS # 5

(1) Massive ring boundary condition and normal modes :-



(a) Equation of motion for the ring :
To this end, we draw the free body diagram of the ring. As the ring is sliding on a frictionless rod, the normal force (N) ~~on~~ on the ring that counteracts the string tension (τ) is along the x-axis.

Free body diagram of the ring :-



Next we can write down the equation of motion:

$$\sum F_x = 0 \Rightarrow N - \tau = 0$$

$$\Rightarrow N = \tau$$

(we used the fact the ~~st~~ ring does not move in the x-direction $\Rightarrow a_x = 0$)
where a_x is the acceleration in the x-direction

$$\sum F_y = M a_y \Rightarrow \boxed{-\tau\left(\frac{\partial y}{\partial x}\right)\bigg|_{x=L} = M \frac{\partial^2 y}{\partial t^2}\bigg|_{x=L}} \quad \text{— (A)}$$

where the derivatives are evaluated at $x = L$.

(b) $M \to 0$.

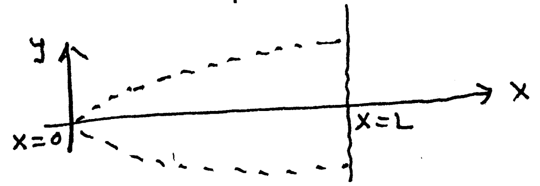
Under this condition, eqn (A) reduced to the condition

$$\frac{\partial y}{\partial x}\bigg|_{x=L} = 0$$

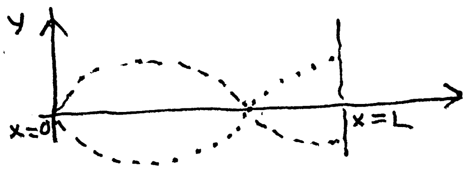
This is the FREE boundary condition.

Allowed wave vectors:

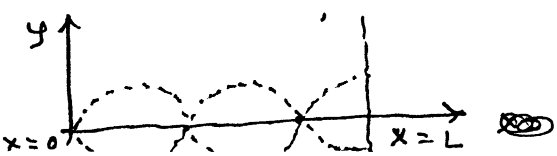
We expect the following progression of standing wave normal modes:



$$\Rightarrow L = \frac{\lambda}{4} \Rightarrow K = \frac{2\pi}{\lambda} = \frac{2\pi}{4L}$$



$$\Rightarrow L = \frac{3\lambda}{4} \Rightarrow K = \frac{2\pi}{\lambda} = 3 \times \frac{2\pi}{4L}$$



$$\Rightarrow L = \frac{5\lambda}{4} \Rightarrow K = \frac{2\pi}{\lambda} = 5 \times \frac{2\pi}{4L}$$

So, in general, the allowed wave vectors for the n^{th} normal mode are:

$$K_n = (2n-1) \times \frac{2\pi}{4L} = \frac{(2n-1)}{2}\frac{\pi}{L} \quad . \quad n = 1, 2, 3, \ldots$$

(c) $M \to \infty$

Let's rewrite eqn. (A) here:

$$-T\left(\frac{\partial y}{\partial x}\right)\Bigg|_{x=L} = M \frac{\partial^2 y}{\partial t^2}\Bigg|_{x=L}$$

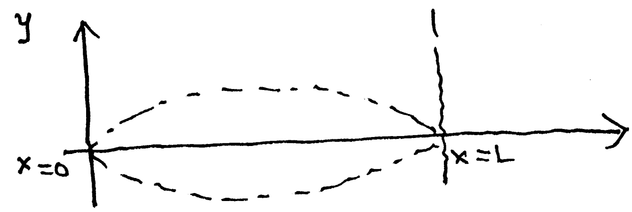
We expect the left hand side of this equation to be finite. So as $M \to \infty$, $\left.\frac{\partial^2 y}{\partial t^2}\right|_{x=L} \longrightarrow 0$. If we start the experiment with the ring at rest at $x = L$ & $y = 0$, it will have $y = 0$ at all future times.

$$\therefore \quad y(L, t) = 0$$

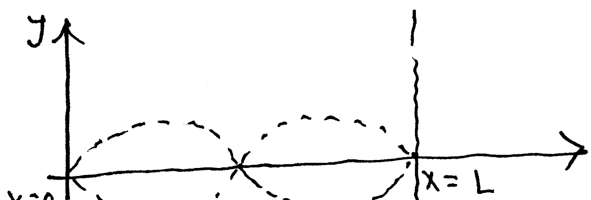
This is the FIXED boundary condition.

Allowed wave vectors:

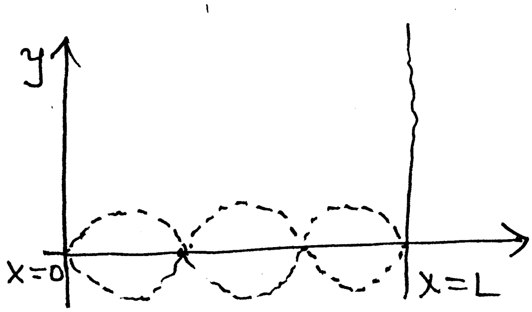
We expect the following progression of normal modes:



$$\Longrightarrow L = \frac{\lambda}{2} \Rightarrow K = \frac{2\pi}{\lambda} = \frac{\pi}{L}$$



$$\Longrightarrow L = 2 \times \frac{\lambda}{2} \Rightarrow K = \frac{2\pi}{\lambda} = \frac{2\pi}{L}$$



$$\Longrightarrow L = 3 \times \frac{\lambda}{2} \Rightarrow K = \frac{2\pi}{\lambda} = \frac{3\pi}{L}$$

So, the allowed wave vectors for the n^{th} normal mode are :

$$K_n = \frac{n\pi}{L} \quad . \quad n = 1, 2, 3, \ldots$$

(d) <u>General Boundary Condition</u> (this states the boundary condition for the intermediate case of finite M, and, as we shall see, also reduces to the free & fixed conditions at appropriate limits of M) :

$$y(x,t) = A \sin(Kx) \cos(\omega t)$$

Plugging this into eqn (A) gives us :

$$-T \times K \times A \cos(KL) \cos\omega t = -M\omega^2 A \sin(KL) \cos\omega t$$

$$\Rightarrow \quad TK = M\omega^2 \tan(KL)$$

$$\Rightarrow \quad \frac{M}{\mu} K = \{\tan(KL)\}^{-1} = \cot(KL) \quad \left(\begin{array}{c}\text{where we use} \\ \omega = c \cdot K \\ \& \ c = \sqrt{\frac{T}{\mu}}\end{array}\right)$$

$$\Rightarrow \quad \boxed{\frac{M}{\mu} K = \cot(KL)}$$

This is the condition that must be true. (B)

(e) Limit $M \longrightarrow 0$

Under this limit, eqn (B) reduces to :

$$\cot(KL) = 0 \quad \Rightarrow \quad K = \frac{\pi}{2L}, \frac{3\pi}{2L}, \frac{5\pi}{2L}, \ldots$$

Notice that we keep only positive values of K as the negative ones do not make physical sense. These values of K agree with the values we found in part (b).

$$K_n = \frac{(2n-1)}{2}\frac{\pi}{L} \quad . \quad n = 1, 2, 3, \ldots$$

(f) Limit $M \to \infty$.

Under this limit, eqn (B) reduces to:

$$\cot(kL) = \infty \Rightarrow k = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L} \; \ldots$$

Notice the we do not include $k=0$ as that is the trivial situation where there are no waves on the string. These values of k agree with what we concluded in part-(c), viz.

$$k_n = \frac{n\pi}{L} \; . \quad n = 1, 2, 3, \ldots$$

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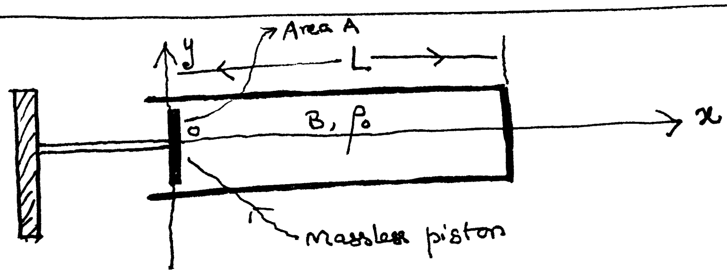
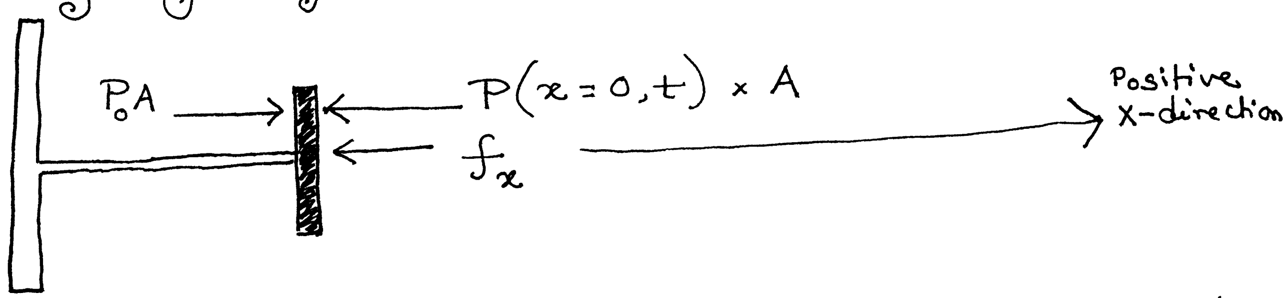
(2) DAMPED BOUNDARY CONDITIONS FOR A SOUND TUBE :-



(a) Free body diagram of the Piston :



$P_0 A$ is the force acting from outside (atmosphere). $P(x=0,t) \times A$ is the force acting from within the cylinder. Assuming that the piston is moving in the positive x-direction, the friction force f_x acts in the negative x-direction. Of course, if the piston were moving in the negative x-direction, f_x would then point in the positive x-direction.

(b) Boundary Condition at the Piston :

As the piston is massless, the equation of motion is :

$$\sum F_x = m a_x = 0$$

$$\Rightarrow \quad \{P_0 - P(x=0,t)\} A + f_x = 0$$

$$\Rightarrow \quad \{P_0 - P(x=0,t)\} A - \beta v_x = 0 \qquad \text{——— (c)}$$

Now we use the constitutive relation and relate the pressure difference $[P_0 - P(x=0,t)]$ to the degree of freedom (the sound speed) $s(x=0,t)$.

To do this we make use of :

$$P_0 - P(x=0,t) = \frac{B \times \Delta V}{V_0} = \frac{B}{A \times dx} \times A \times \{s(dx) - s(0)\}$$

where B is the bulk modulus of air. $s(0)$ is the sound displacement at $x=0$ and $s(dx)$ is the sound displacement at $x = 0+dx$. V_0 is the volume of the chunk of air between $x=0$ and $x=0+dx$.

Clearly then,

$$P_0 - P(x=0,t) = B \left.\frac{\partial s}{\partial x}\right|_{x=0} = B \frac{\partial s(x=0,t)}{\partial x}$$

So eqn (c) becomes :

$$BA \frac{\partial s}{\partial x}(x=0,t) = \beta v_x = \beta \frac{\partial s(x=0,t)}{\partial t}$$

where we used the fact that v_x is the speed of piston which is equal to the sound speed at $x=0$.

So, the equation of motion can written as the following boundary condition :-

$$\boxed{BA \frac{\partial s}{\partial x}(x=0,t) = \beta \frac{\partial s}{\partial t}(x=0,t)} \qquad \text{———} \quad (D)$$

(c) $s(x,t) = S_m \sin(kx+\phi_0)\cos\omega t$

Substituting this standing wave solution in eqn (D), we get :

$$BA \times k S_m \cos(kx+\phi_0)\cos\omega t \Big|_{x=0} = \beta \times -\omega S_m \sin(kx+\phi_0)\sin\omega t \Big|_{x=0}$$

$$\Rightarrow \boxed{BAk = -\beta\omega \tan(kx+\phi_0)\Big|_{x=0} \tan\omega t}$$

$$\Rightarrow \boxed{BAk = -\beta\omega \tan\phi_0 \tan\omega t}$$

This is not satisfied for a nonzero β for all time t.

One can explain this using the principle of conservation of energy. The given standing wave solution has an amplitude that is constant in time. Consequently, the total energy of the standing wave is constant. However, for a nonzero β there will be dissipation of energy due to friction. And this situation cannot be described with the given standing wave solution which has constant energy.

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