Problem set #6

#1 a) The wavelength in vacuum is

$\lambda = 0.20 \text{ nm} = 2 \times 10^{-10} \text{ m}$.

The speed is $c = \lambda f$ so

$$f = \frac{c}{\lambda} = \frac{3 \cdot 10^{8} \text{ m/s}}{2 \cdot 10^{-10} \text{ m}} = 1.5 \times 10^{18} \text{ Hz}$$

where c is the speed of light.

$$\boxed{f = 1.5 \cdot 10^{18} \text{ Hz}}$$

b)

Wave type	frequency (Hz)
radio	$< 10^{9}$
micro	10^{9} to 10^{12}
infrared	10^{12} to $4.3 \cdot 10^{14}$
visible	$(4.3$ to $5.7) \cdot 10^{14}$
ultraviolet	$5.7 \cdot 10^{14}$ to 10^{16}
→ X-ray	10^{16} to 10^{19} ←
γ-ray	$> 10^{19}$

Radiation is in the X-ray region

2

An EM wave in vacuum is discribed by

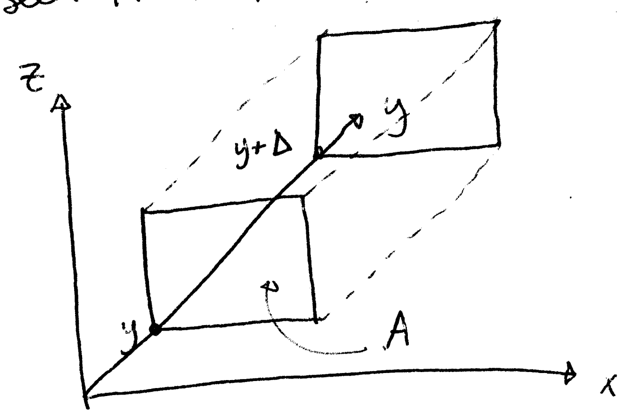
$$\vec{B}(\vec{r},t) = \vec{B}_o \sin(ky - \omega t)$$

wave is propagating in y-dir

a) Starting from

$\oiint \vec{B} \cdot d\vec{A} = 0$, let's show that $B_{oy} = 0$

As seen in lecture, we construct a box,



$$\oiint \vec{B} \cdot d\vec{A} = \iint_{top} \vec{B} \cdot d\vec{A} + \iint_{bottom} \vec{B} \cdot d\vec{A} + \iint_{left} \vec{B} \cdot d\vec{A} + \iint_{right} \vec{B} \cdot d\vec{A}$$

$$+ \iint_{front} \vec{B} \cdot d\vec{A} + \iint_{back} \vec{B} \cdot d\vec{A}$$

Assuming we have plane waves, for each value of y, $B_{top} = B_{bottom} = B_{left} = B_{right}$,

however, $d\vec{A}_{top} = -d\vec{A}_{bottom}$ and $d\vec{A}_{left} = -d\vec{A}_{right}$

#2 a) hence,

$$\iint_{top} \vec{B}\cdot d\vec{A} + \iint_{bottom} \vec{B}\cdot d\vec{A} = 0$$

$$\iint_{left} \vec{B}\cdot d\vec{A} + \iint_{right} \vec{B}\cdot d\vec{A} = 0$$

Leaving,

$$0 = \iint_{front} \vec{B}\cdot d\vec{A} + \iint_{back} \vec{B}\cdot d\vec{A}$$

but $\vec{B}$ is constant on these plane,

$$= -A \cdot B_y(y) + A \cdot B_y(y+\Delta)$$

since $d\vec{A}_{front} = -A\hat{y}$

$d\vec{A}_{back} = A\hat{y}$

This means that

$$0 = A\left(B_y(y+\Delta) - B_y(y)\right)$$ for all Δ.

which imply that $B_y(y) = 0$

since $0 = B_y(y) = B_{oy} \sin(ky - \omega t)$,

we have that $\boxed{B_{oy} = 0}$

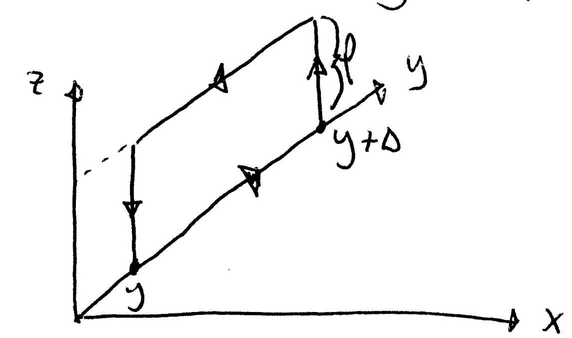
#2

b) We want to find a relationship between

$\frac{\partial B_x}{\partial t}$ and $\frac{\partial E_z}{\partial y}$.

This time, we have to use $\oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint \vec{B}\cdot d\vec{A}$

and the following loop:



(loop is in the yz-plane)

$$\int_{front} \vec{E}\cdot d\vec{l} + \int_{back} \vec{E}\cdot d\vec{l} + \int_{top} \vec{E}\cdot d\vec{l} + \int_{bottom} \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint \vec{B}\cdot d\vec{A}$$

but the "top" and "bottom" cancel because for each value of y, their $\vec{E}(y)$ are equal but their $d\vec{l}$ are opposite.

Also, $\vec{E}(y)$ = constant on the front and back loop.

$$-\ell\, E_z(y) + \ell\, E_z(y+\Delta) = \frac{-\partial}{\partial t}\int B_x \cdot dA$$

#2 b) which gives, after dividing by the area ($A = \ell \cdot \Delta$)

$$\frac{E_z(y+\Delta) - E_z(y)}{\Delta} = -\frac{\partial}{\partial t} \iint \frac{B_x \cdot dA}{A}$$

taking the limit as $\Delta \to 0$

$$\frac{\partial E_z(y)}{\partial y} = -\frac{\partial}{\partial t} \bar{B}_x = -\frac{\partial}{\partial t} B_x(y)$$

using $\iint \frac{B_x \, dA}{A} = \bar{B}_x$ → the average

$= B_x(y)$ in the limit $\Delta \to 0$

so $$\boxed{\frac{\partial E_z(y)}{\partial y} = -\frac{\partial}{\partial t} B_x(y)}$$

and $$\frac{\partial E_z(y)}{\partial y} = -\frac{\partial}{\partial t} B_x(y) = \omega B_{0x} \cos(ky - \omega t)$$

so $$E_z(y) = \int \omega B_{0x} \cos(ky - \omega t)\, dy$$

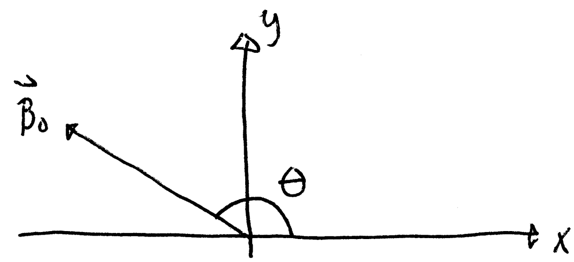
giving: $$\boxed{E_z(y,t) = \frac{\omega}{k} B_{0x} \sin(ky - \omega t)}$$

with $\frac{\omega}{k} = c$, the speed of light.

3.

We have an EM plane wave in vacuum described by $\vec{B}(\vec{r},t) = \vec{B}_0 \cos(\omega t + kz + \pi/12)$

with $k = 3.0 \frac{rad}{m}$, $|\vec{B}_0| = 5.0 \cdot 10^{-10}$ Tesla and $\vec{B}_0$ oriented such that $\theta = 143°$ in the following sketch:



a) Since the argument of the cosine function of $\vec{B}(\vec{r},t)$ depends only on "z" and "t", and since the relative sign between kz and ωt is positive, the EM wave propagates in the $-\hat{z}$ direction.

b) For a wave propagating in the $\pm\hat{z}$-direction we have
$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \quad \text{and} \quad \frac{\partial E_y}{\partial z} = +\frac{\partial B_x}{\partial t}.$$

* 3 b) Cont.

Assuming $\vec{E}(\vec{r},t) = \vec{E}_0 \cos(\omega t + kz + \pi/12)$ we can replace $\vec{E}$ and $\vec{B}$ in the previous equations:

$$\frac{\partial E_x}{\partial z} = \frac{-\partial B_y}{\partial t}$$

$$-k E_{ox} \sin(\cancel{\omega t + kz + \pi/12}) = + \omega B_{oy} \sin(\cancel{\omega t + kz + \pi/12})$$

$$E_{ox} = -\frac{\omega}{k} B_{oy} = -c B_{oy} .$$

Doing the same with the second equation we get something really similar with an extra minus sign, so

$$\underline{E_{ox} = -cB_{oy}} \quad \text{and} \quad \underline{E_{oy} = c B_{ox}} .$$

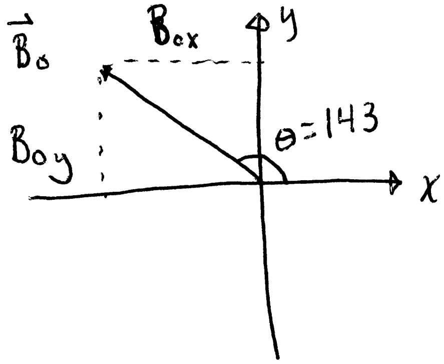
Since $E_{oz} = 0$ (in the direction of propagation)

$$|\vec{E}_0| = \sqrt{E_{ox}^2 + E_{oy}^2 + 0} = \sqrt{c^2 B_{ox}^2 + c^2 B_{oy}^2} = c\sqrt{B_{ox}^2 + B_{oy}^2}$$

so $\boxed{|\vec{E}_0| = c\,|\vec{B}_0|}$

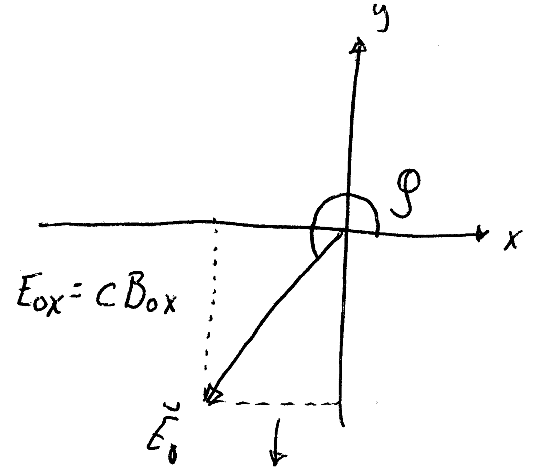
3 b) cont.

As for the direction,



$E_{oy} = cB_{ox}$

$E_{ox} = -cB_{oy}$



$|E_{oy}| = |-cB_{oy}|$

Here, we can verify that $\vec{E} \times \vec{B}$ is in the propagation-dir

Since $\vec{B} \perp \vec{E}$, we have

$$\varphi = \theta + 90^\circ = 143^\circ + 90^\circ = 233^\circ$$

so $\boxed{|\vec{E}_o| = c|\vec{B}_o| \text{ and the angle describing } \vec{E}_o \text{ in polar coordinate is } \varphi = 233^\circ}$

c) The frequency is $f = \frac{\omega}{2\pi}$ and $\omega = ck$

so $$f = \frac{ck}{2\pi} = \frac{3 \cdot 10^8 \, m/s \cdot 3.0 \, \frac{rad}{m}}{2\pi} = 1.432 \cdot 10^8 \, Hz$$

and $$\lambda = \frac{2\pi}{k} = \frac{2\pi}{3.0 \, \frac{rad}{m}} = 2.094 \, m$$

so $\boxed{f = 1.432 \cdot 10^8 \, Hz \text{ and } \lambda = 2.094 \, m}$

*3 d)

To detect the EM wave, we need to have the highest value of induced emf possible.

Since $\mathcal{E}_{ind} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\int \vec{B}\cdot d\vec{A}\right)$,

this will be maximum when $\vec{B}$ is parallel to $d\vec{A}$.

- Because the direction of $d\vec{A}$ correspond to the direction of the axis of the coil, we want to have

the loop axis in the $\vec{B}$ direction.

- We have $\int \vec{B}\cdot d\vec{A} = \int |B_0| \cos(\omega t + kz + \pi/12)\, dA$

and we are integrating on a circle centered in z of area $A = \pi r^2 = \pi\cdot(0.02\,m)^2 = 4\pi\cdot 10^{-4}\,m^2$.

- Since the ratio $\frac{r}{\lambda} = \frac{0.02\,m}{2.09\,m} \sim 0.01$ is small,

we can assume that $\vec{B}(z,t)$ is constant over the coil.

* 3 d) cont.

Then $\int \vec{B}(z,t) \cdot d\vec{A} = A \cdot B(z,t)$

and $\mathcal{E}_{ind}(z,t) = \frac{-d}{dt}\left(|B_0| \pi r^2 \cos(\omega t + kz + \pi/12)\right)$

$= \omega |B_0| \pi r^2 \sin(\omega t + kz + \pi/12)$

and the maximum value is

$\mathcal{E}_{ind,max} = \omega |B_0| \pi r^2 = 2\pi \cdot 1.432 \cdot 10^8 Hz \cdot 5 \cdot 10^{-10} T \cdot \pi \cdot (0.02 m)^2$

$\boxed{\mathcal{E}_{ind,max} = 5.6 \times 10^{-4} \ V}$ $\quad 1 V = 1 \frac{T \cdot m^2}{s}$

e) Increase r will increase $\mathcal{E}_{ind,max}$.

However, at some point, we won't be able to consider $\vec{B}(z,t) =$ const. on the coil.

If $r \sim \lambda$ we will get a lot of cancellation in $\int \vec{B} \cdot d\vec{A}$ on the coil, provoking a decrease in $\mathcal{E}_{ind,max}$.

Therefore we want to increase r and keep the relation $r \ll \lambda$.