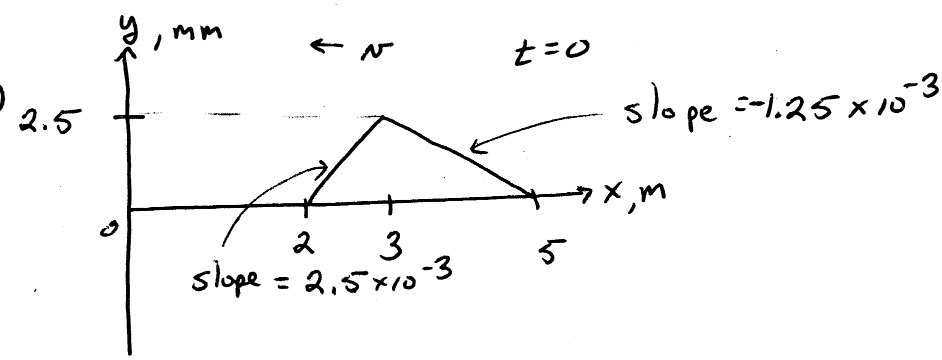
Physics 214 Problem Set #7 Solutions

1



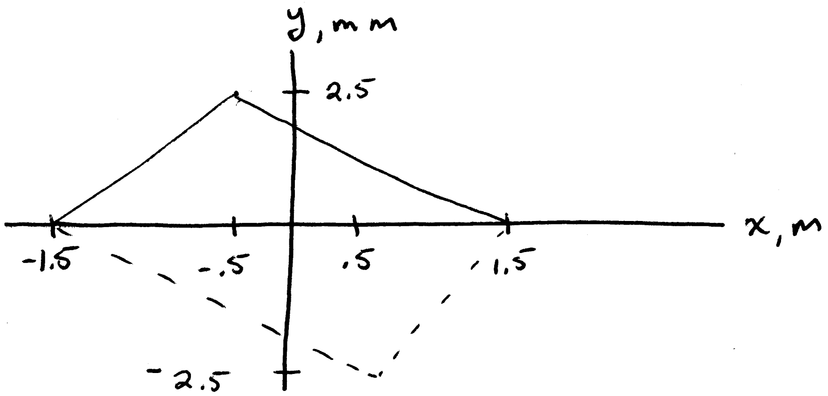
(a) fixed B.C. at $x = 0$

$t = 0.035$

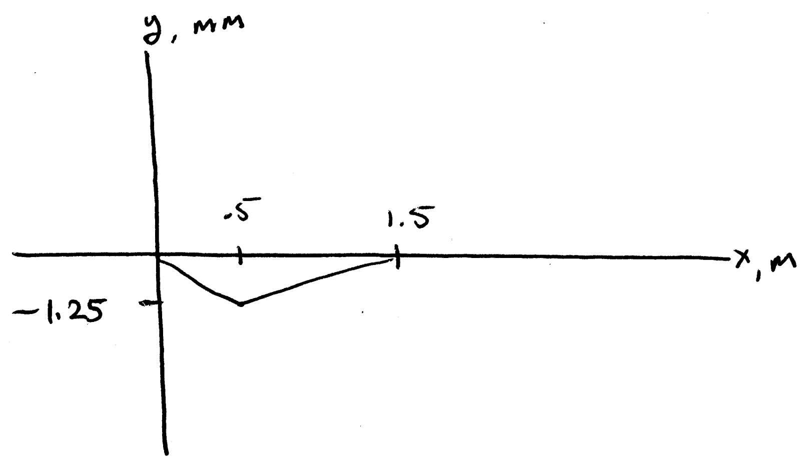
$\Delta x = 3.5$ m

— incident wave

- - - reflected wave

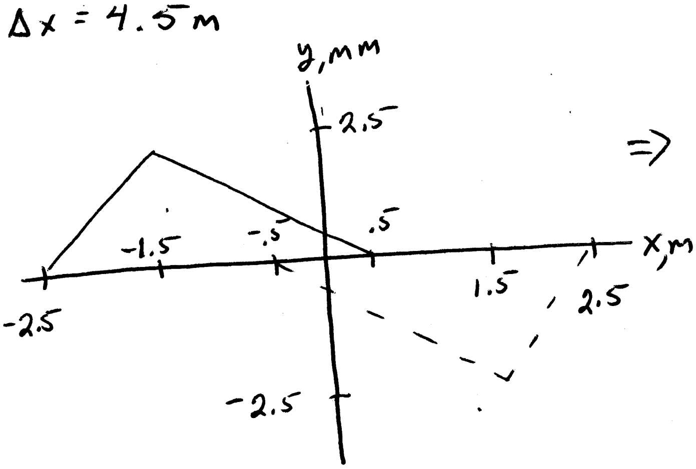


$\Rightarrow$

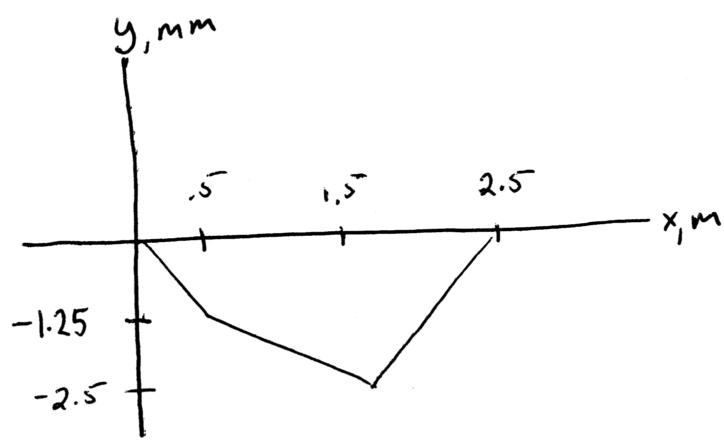


$t = 0.045$

$\Delta x = 4.5$ m



$\Rightarrow$

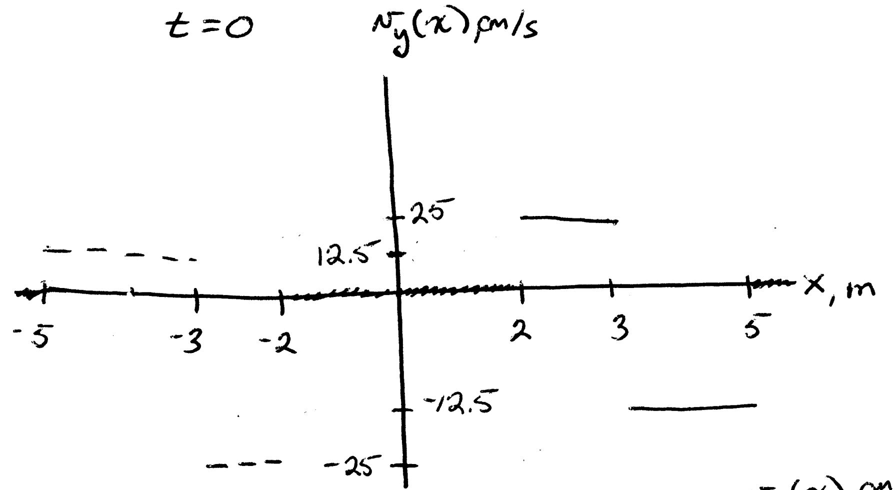


(b) Even though this problem has waves with sharp points, we know that a "physical" wave in the real world won't have infinitly sharp features. We'll ignore the discontinuities in the velocity sketches. $v_y(x) = \frac{\partial y}{\partial t} = \frac{\partial y}{\partial x}\frac{dx}{dt} = \mp v \frac{\partial y}{\partial x}$ for either of the traveling wave components. Then add velocities since $\frac{\partial}{\partial t}(y_1 + y_2) = \frac{\partial y_1}{\partial t} + \frac{\partial y_2}{\partial t}$

$t = 0$

— velocity of incident

--- velocity of reflected



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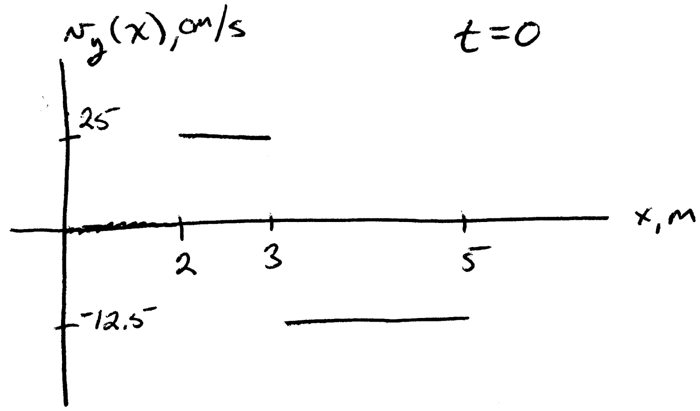
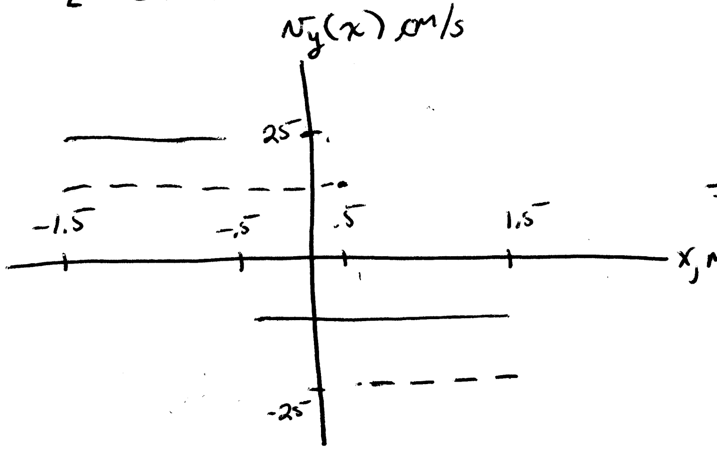
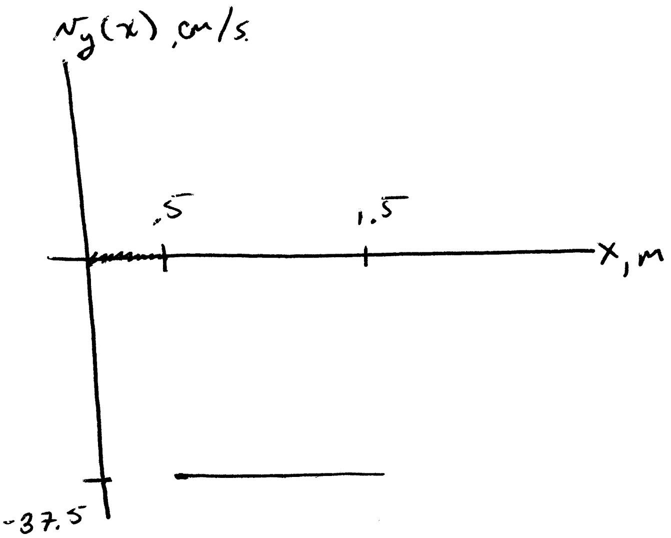


$t = 0.035$



$\Rightarrow$



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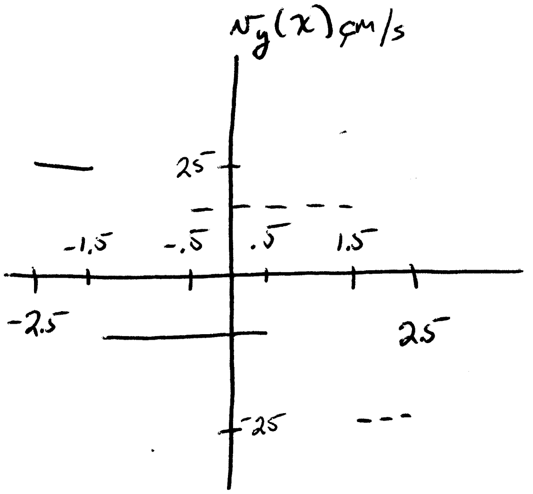
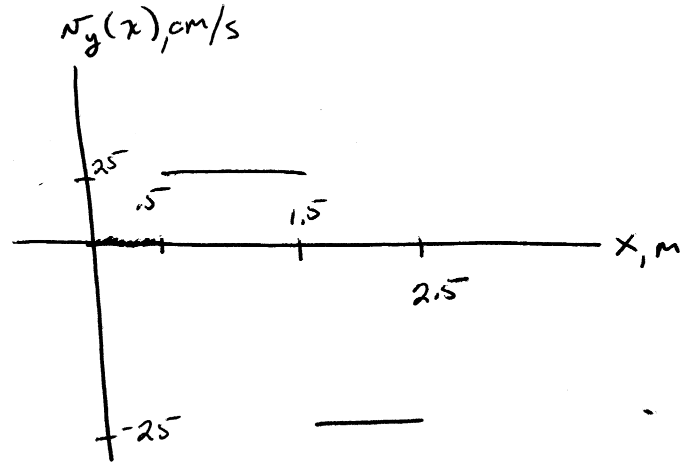
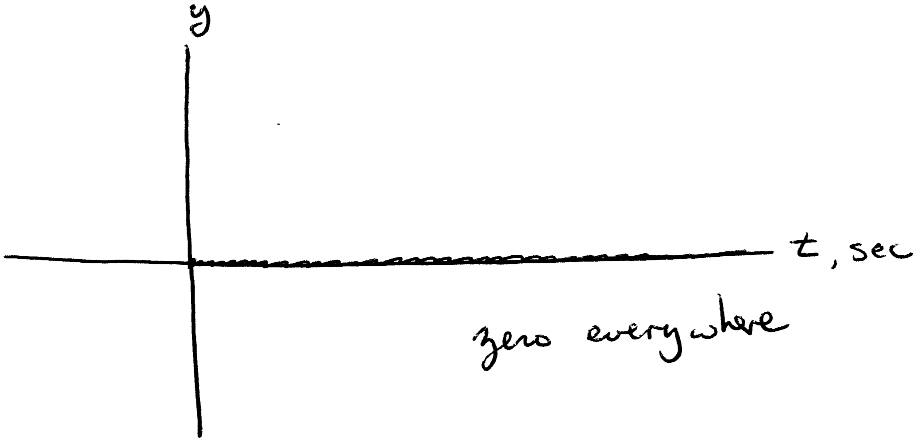


$\Rightarrow$



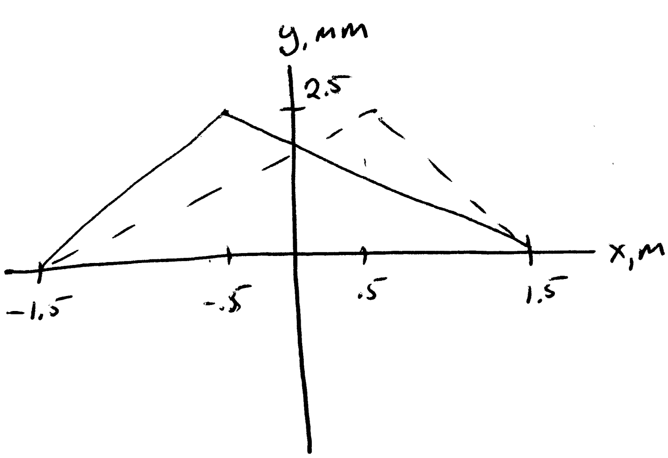
(c) at $x = 0$ with a fixed B.C.



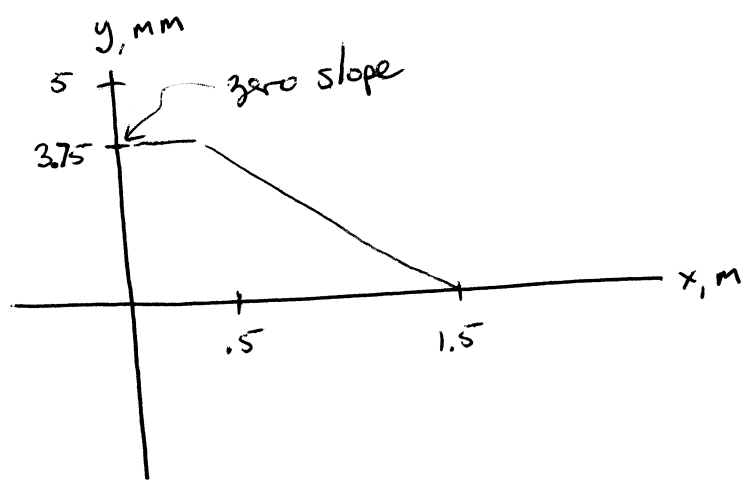
(d) free B.C. at $x = 0$

$t = 0.035$

$\Delta x = 3.5$ m

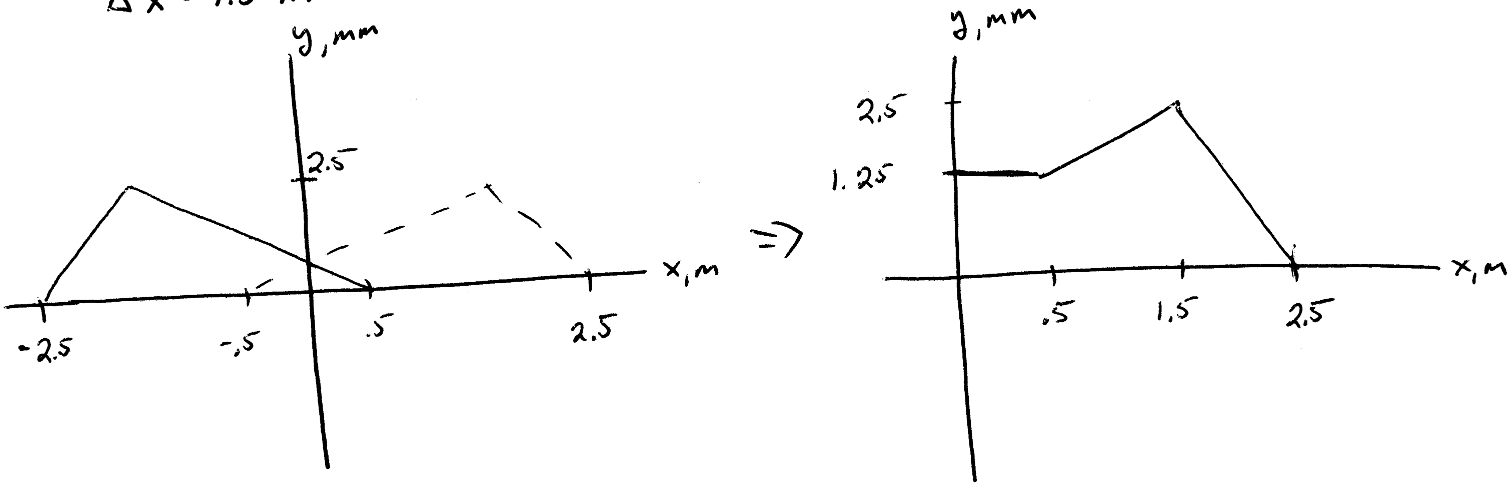


$\Rightarrow$

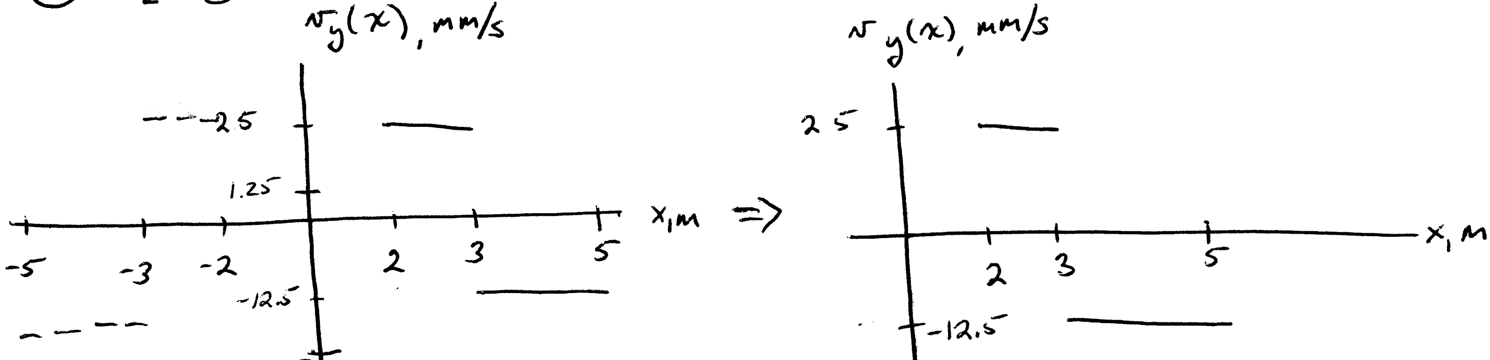


$t = 0.045$

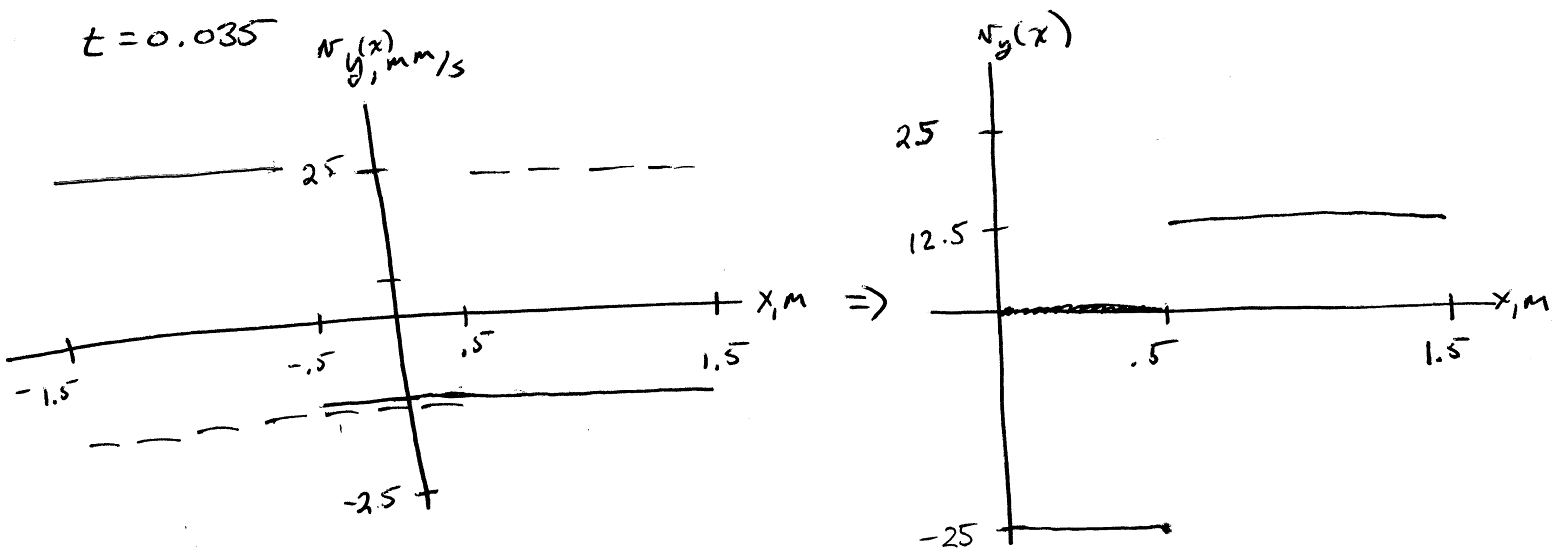
$\Delta x = 4.5 \text{ m}$



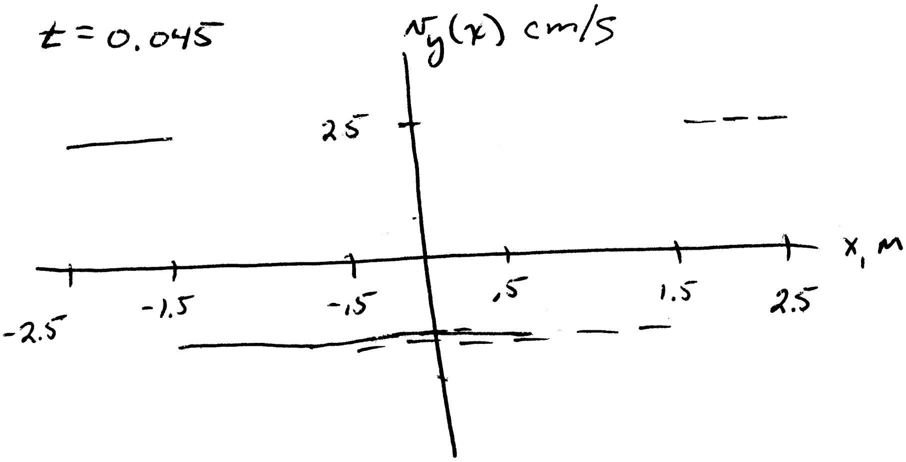
(e) $t = 0$

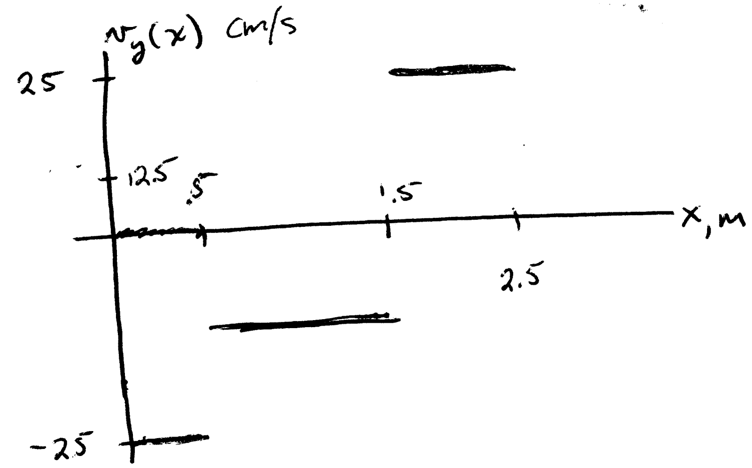


$t = 0.035$

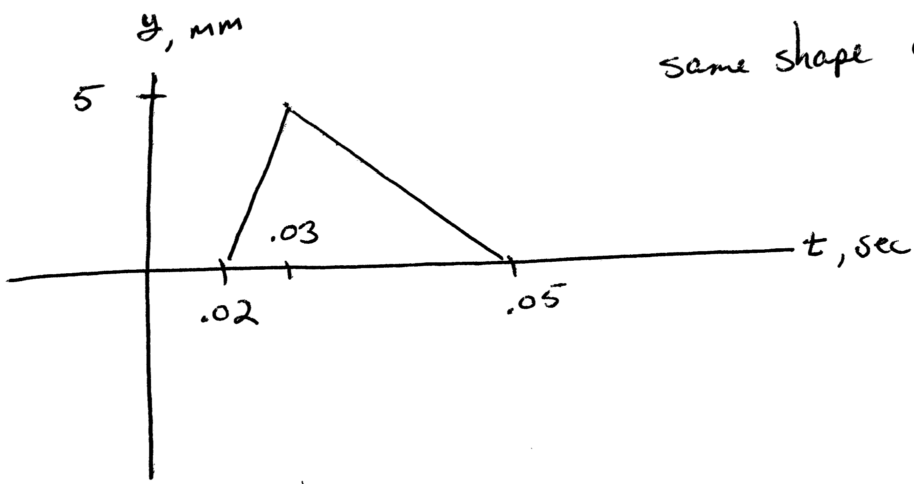


$t = 0.045$





(f) at $x = 0$ with a free B.C.



same shape as original wave in real space but twice as tall.

(2)

(a) $s(x,t) = A \underbrace{\cos(kx)} \sin(\omega t)$

This part says that the displacement at $x=0$ is an antinode and a pressure node. This is true for an open end at $x=0$.

Since it's open, we require $\left.\dfrac{\partial s(x,t)}{\partial x}\right|_{x=0} = 0$

(b) $$s(x,t) = A\left(\frac{e^{ikx} + e^{-ikx}}{2}\right)\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right)$$

$$= \frac{A}{4i}\left(e^{i(kx+\omega t)} - e^{i(kx-\omega t)} + e^{i(-kx+\omega t)} - e^{-i(kx+\omega t)}\right)$$

$$= \frac{A}{4i}\left(\underbrace{e^{i(kx+\omega t)} - e^{-i(kx+\omega t)}}_{\Downarrow} + \underbrace{e^{-i(kx-\omega t)} - e^{i(kx-\omega t)}}_{\Downarrow}\right)$$

$$= \frac{A}{2}\left(\sin(kx+\omega t) - \sin(kx-\omega t)\right) \qquad \omega = vk$$

$$= \frac{A}{2}\left(\underbrace{\sin[k(x+vt)]}_{g(x+vt)} - \underbrace{\sin[k(x-vt)]}_{f(x-vt)}\right)$$

$g(u) = \frac{A}{2}\sin ku \qquad f(u) = -\frac{A}{2}\sin ku$

(e) Show $f(-u) = +g(u)$ as required for an open end at $x=0$.

$$f(-vt) \stackrel{?}{=} +g(vt) \qquad \text{let } u = vt$$

$$f(-u) \stackrel{?}{=} +g(u)$$

$$-\frac{A}{2}\sin(K(-u)) \stackrel{?}{=} +\frac{A}{2}\sin(K(u)) \checkmark \text{ from part (b)}$$

(3) $y(x,t) = f(x - vt) + g(x + vt)$

(a) $v_y(x, t=0) = 0$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0 = \left.\frac{\partial f}{\partial t}\right|_{t=0} + \left.\frac{\partial g}{\partial t}\right|_{t=0} = f'(x)(-v) + g'(x)v$$

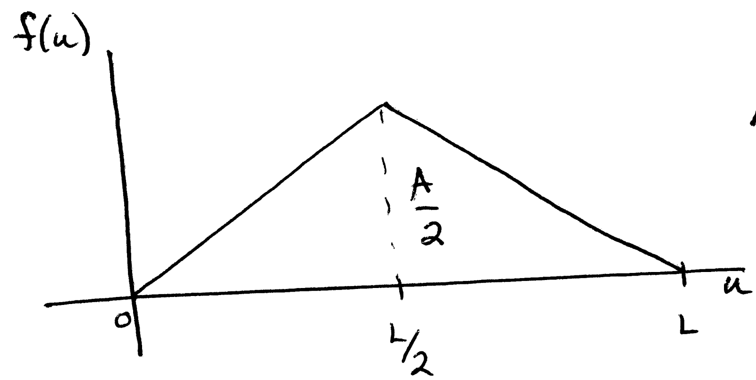
$$\Rightarrow f'(x) = g'(x)$$

$f(x) = g(x) + C \qquad C = 0$ using $g(\pm\infty) = f(\pm\infty) = 0$

$$f(x) = g(x)$$

$$\boxed{\text{or } f(u) = g(u)}$$

(b)



A is the initial amplitude of the plucked string

(c) at $x=0$:

$f(x - vt) = -g(x + vt)$ since $y(x=0,t) = 0$

$f(-vt) = -f(vt)$ using results from (a)

$$\boxed{f(u) = -f(-u)} \qquad -u = vt$$

This says that $f(u)$ looks the same after reflection in the x and y axes.

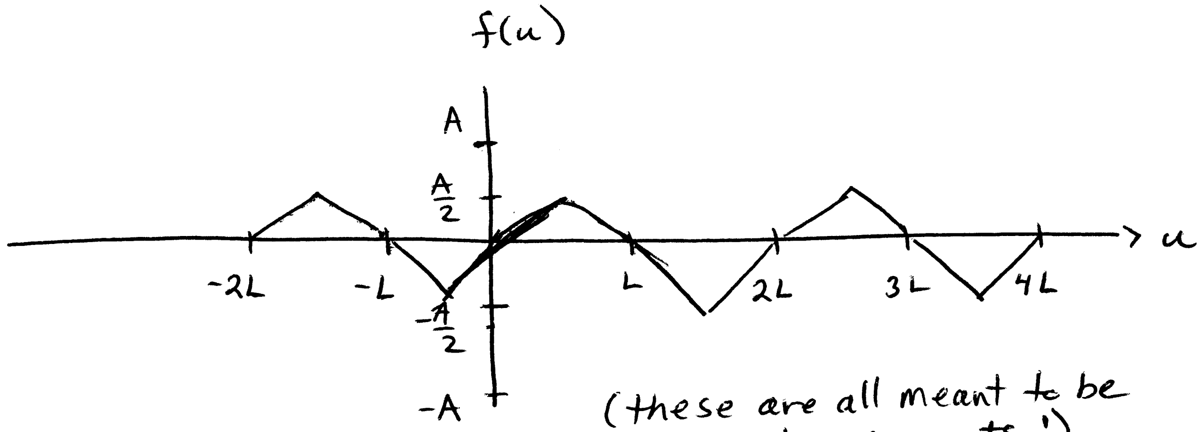
(d) at $x=L$:

$$y(L,t) = 0 = f(L-vt) + g(L+vt)$$

$$f(L-vt) = -f(L+vt)$$ using results from (a)

$$\boxed{f(L-u) = -f(L+u)}$$ let $u = vt$

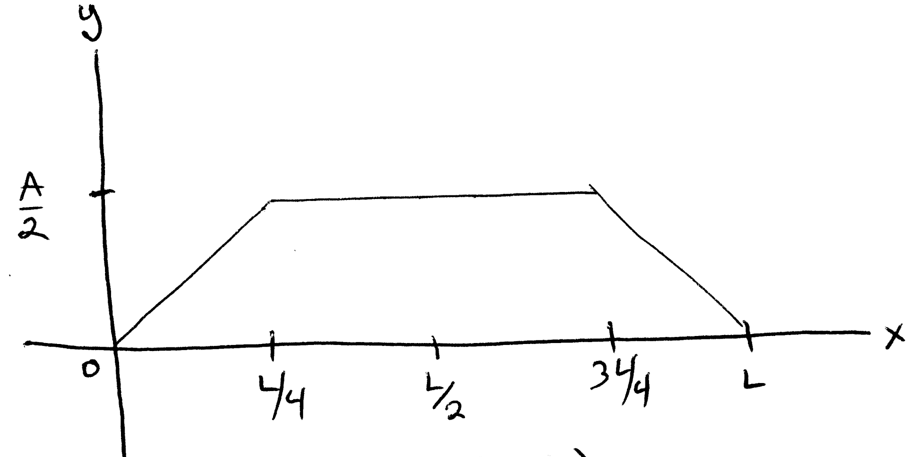
(e)



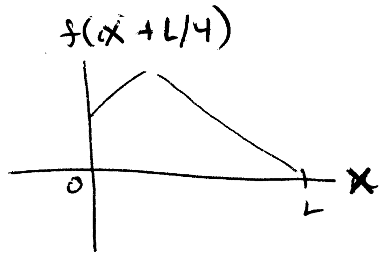
(these are all meant to be straight line segments!)

(f) $t = \frac{L}{4v}$

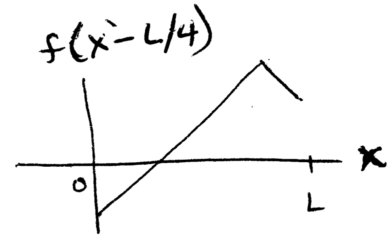
$$y\left(x, \frac{L}{4v}\right) = f\left(x + \frac{L}{4}\right) + f\left(x - \frac{L}{4}\right)$$



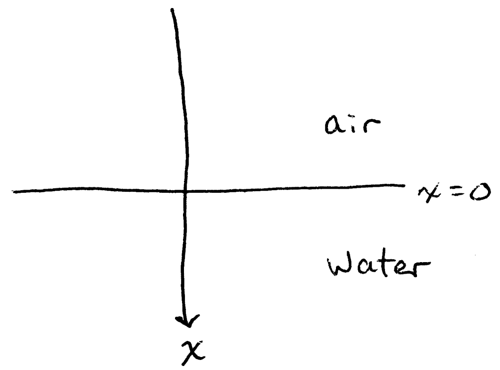
This comes from adding:



and



(4)



$v_{air} = 334$ m/sec

$v_{water} = 1480$ m/sec

$$s(x,t) = A\cos(k_a x - \omega t) + B\cos(k_a x + \omega t) \qquad x < 0$$

$$s(x,t) = C\cos(k_w x - \omega t) \qquad x > 0$$

At the boundary, require continuity:

$$s(x<0,t) = s(x>0,t) \quad \text{at } x=0$$

$$A + B = C$$

Require, also, force balance:

$$B_a \left.\frac{\partial s(x<0,t)}{\partial x}\right|_{x=0} = B_w \left.\frac{\partial s(x>0,t)}{\partial x}\right|_{x=0}$$

where B_a, B_w are the Bulk modulus for air and bulk modulus for water

$$-AB_a k_a + BB_a k_a = -CB_w k_w$$

$$k = \frac{\omega}{v}$$

$$-A\frac{B_a}{v_a} + B\frac{B_a}{v_a} = -\frac{CB_w}{v_w}$$

Solve for $\frac{C}{A}$ to get transmission amplitude in units of initial amplitude.

$$B = C - A$$

$$-A\frac{B_a}{v_a} + (C-A)\frac{B_a}{v_a} = -C\frac{B_w}{v_w}$$

$$-2A\frac{B_a}{v_a} = -C\left(\frac{B_w}{v_w} + \frac{B_a}{v_a}\right)$$

$$\frac{C}{A} = \frac{2\,B_a/v_a}{\frac{B_w}{v_w} + \frac{B_a}{v_a}}$$

since $\frac{B}{v} = Z$, this agrees with the course notes.

Easier to look up densities than bulk moduli, so express Z in terms of ρ and v: (use $v^2 = B/\rho$)

$$Z = \frac{B}{v} = \frac{\rho v^2}{v} = \rho v$$

$$Z_a = \rho_a v_a = 1.2\ \mathrm{kg/m^3} \cdot 334\ \mathrm{m/s} \simeq 400\ \mathrm{kg \cdot m^{-2} s^{-1}}$$

$$Z_w = \rho_w v_w = 1000\ \mathrm{kg/m^3} \cdot 1480\ \mathrm{m/s} \approx 1.48 \times 10^6\ \mathrm{kg \cdot m^{-2} \cdot s^{-1}}$$

$$\frac{C}{A} = \frac{2Z_a}{Z_w + Z_a} \approx \frac{800}{1.48 \times 10^6} = 5.5 \times 10^{-4}$$

Amplitude of transmitted wave is 5.5×10^{-4} times amplitude of incident wave; almost total reflection.