Cornell University

Department of Physics

Phys 214 October 22, 2004

Waves, Optics, and Particles, Fall 2004

Homework Assignment # 8

(Due Thursday, October 28 at 5:00pm *sharp.*)

Agenda and readings for the week of October 25:

Skills to be mastered:

- Be able to use complex wave amplitudes;
- Understand the relation between wave intensity and its complex amplitude;
- Understand the superposition principle for complex wave amplitudes;
- Understand the derivation of the interference pattern from two or more narrow slits;
- Be able to determine the properties of a slit system that produces a known interference pattern;
- Be able to use phasors to illustrate interference.

Lectures and Readings:

- Lec 17, 10/26 (Tue): N-slit interference pattern
  Readings: LN "Wave Phenomena II: Interference," Sec. 3.3; VW pp. 284-288.
- Lec 18, 10/28 (Thu): Finite-slit interference
  Readings: LN "Wave Phenomena II: Interference," Sec. 4; VW pp. 288-298.

1 Michelson Interferometer with Microwaves

Figure 1 shows a Michelson interferometer similar to one constructed in Lab Experiment 2. (T) is the transmitter and (R) is the receiver. The full reflectors (1) and (2) are at the ends of arm (1) and arm (2) of the interferometer, a partial reflector is at the center. The beams from the two spectrometer arms interfere on their way from the partial reflector to the receiver. Reflector (1) is being moved in the course of the experiment, its position is X. Take the speed of microwaves in air to be $c = 3 \times 10^8$ m/s.

(a) As you move reflector (1) along arm (1), your receiver (R) registers maxima of intensity at many reflector positions X, and you measure five of these positions, namely at $X = 20.0$ cm, 23.5 cm, 26.5 cm, 30.0 cm, and 33.4 cm. What is the wavelength of the microwaves?

(b) Suppose you have your interferometer set up such that the path lengths in arm (1) and arm (2) are equal and the phase difference between the two waves arriving at the detector is zero. You now insert a sheet of polyethylene with index of refraction $n = 1.6$ into arm (1), between the central partial reflector and reflector (1). The effective width of the sheet is 0.60 cm. (In other words, the length of the path that the microwaves take through the sheet is 0.60 cm.) You leave the transmitter, receiver, and both eflectors in place. Note that, while the frequency is unchanged, the speed of light in the polyethylene is c/n, and the wave vector is changed accordingly. Calculate the phase difference $\Delta\phi$ between the two waves that has been introduced by inserting the polyethylene slab.

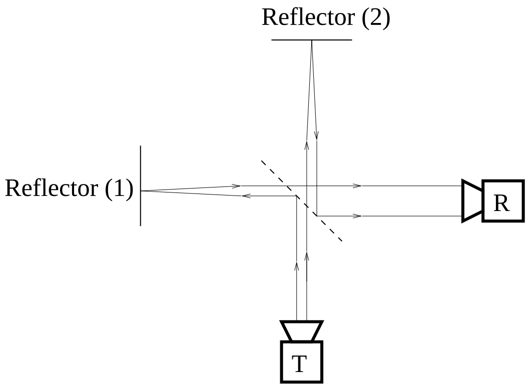

Figure 1: A Michelson interferometer

(c) You want to move reflector (1) so as to make the phase difference zero again (in other words: you want to make the optical paths in the two spectrometer arms equal again). In what direction do you move reflector (1)? (Towards the partial reflector at the center or away from the partial reflector at the center?) *Explain.*

(d) Calculate the distance ΔX that you need to move reflector (1) in order to reestablish zero phase difference. *Show* how you arrive at your result.

2 Impedance Matching

In this problem, you will consider reflection for waves on a string. The string has mass per unit length μ, applied tension τ, and impedance $z = \sqrt{\tau\mu}$. The string extends from $x = -\infty$ to $x = 0$. At $x = 0$, the string is connected to a massless, frictionless slip ring. The slip ring is connected to a dashpot that exerts a force $f_y = -bv_y$ on the slip ring.

(a) Explain why, for a sinusoidal traveling wave incoming from the left and moving to the right, the solution to the wave equation has the form

$$y(x \leq 0, t) = \Re\left[\underline{A}e^{-i\omega t}\left(e^{ikx} + \underline{R}e^{-ikx}\right)\right]$$

Your explanation can be simply the identification of the meaning of each term in each equation.

(b) Apply $\sum \vec{F} = m\vec{a}$ to the slip ring to find the boundary condition for $x = 0$. (Your answer should be in the form of a differential equation that must hold at $x = 0$.)

(c) Use the boundary condition equation along with the form of the solution to find $\underline{R}$ in terms of b and z *only.*

(d) For what value(s) of b is there no reflected wave? Explain the term "impedance matching."

3 Reflection *and Transmission* with an Attached Dashpot

In this problem, you will consider transmission and reflection for waves on a string. The string has mass per unit length μ, applied tension τ, and impedance $z = \sqrt{\tau\mu}$ and may be regarded as infinitely long ($x = -\infty$ to $x = +\infty$). However, a dashpot attached to the string at $x = 0$ creates a disruption to normal propagation of waves past that point. The dashpot exerts an external force $f_y = -bv_y$ on the point $x = 0$. (This is just like the previous problem except that string is tied to *both* sides of the slip ring. Thus, if a wave incoming from the left and moving to the right reaches $x = 0$, we have the possibility of transmission in addition to reflection.)

(a) Explain why, for a sinusoidal traveling wave incoming from the left and moving to the right, the solution to the wave equation has the form

$$y_0(x \leq 0, t) = \Re\left[\underline{A}e^{-i\omega t}\left(e^{ikx} + \underline{R}e^{-ikx}\right)\right]$$

$$y_1(x \geq 0, t) = \Re\left[\underline{A}e^{-i\omega t}\left(\underline{T}e^{ikx}\right)\right].$$

Your explanation can be simply the identification of the meaning of each term in each equation.

(b) Explain why the following boundary condition must hold at the point $x = 0$:

$$y_0(x = 0, t) = y_1(x = 0, t)$$

(c) Apply $\sum \vec{F} = m\vec{a}$ to the point $x = 0$ to find another boundary condition equation. (Your answer should be in the form of a differential equation involving both y_0 and y_1 that must hold at $x = 0$.)

(d) Use the two boundary condition equations along with the form of the solution to find two equations for the unknowns $\underline{R}$ and $\underline{T}$. Solve these equations to find $\underline{R}$ and $\underline{T}$ in terms of b and z *only*.

(e) Calculate $\underline{R}$ and $\underline{T}$ in the limiting case $b \to 0$. Explain why this result is sensible.

(f) Calculate $\underline{R}$ and $\underline{T}$ in the limiting case $b \to \infty$. Explain why this result is sensible.

4 Transmission and Reflection in Strings (II)

A thin string (mass per unit length μ_0, wave speed c_0) of length a is attached to a wall at $x = 0$. The other end of the string is attached to a thicker string (mass per unit length μ_1, wave speed c_1), see Fig. 2. A sinusoidal wave of amplitude A and wavevector k_1 travels in from the right along the thick string and is then reflected. Using complex representation, the motion of the string can be described by

$$\begin{aligned} y_0(x < a, t) &= \Re[e^{-i\omega t}(\underline{B}e^{-ik_0x} + \underline{C}e^{ik_0x})], \\ y_1(x > a, t) &= \Re[\underline{A}e^{-i\omega t}(e^{-ik_1(x-a)} + \underline{r}e^{ik_1(x-a)})]. \end{aligned} \tag{1}$$

where $\underline{r}$ is the reflection coefficient. In this problem, you will find $\underline{r}$ in two different ways.

4.1 Matching at the Boundary

The first way involves using boundary conditions at $x = 0$ and $x = a$:

(a) Using the boundary condition at $x = 0$, show that the motion of the light string is described by

$$y_0(x < a, t) = \Re[e^{-i\omega t}\underline{D}\sin(k_0x)]. \tag{2}$$

How is the coefficient $\underline{D}$ related to $\underline{B}$ and $\underline{C}$?

(b) Write down the equations that $y_0(x, t)$ and $y_1(x, t)$ should satisfy at $x = a$. What is the physical meaning of each equation?

(c) Solve these equations to obtain r. Express your answer in terms of c_0, c_1, k_1, and a.

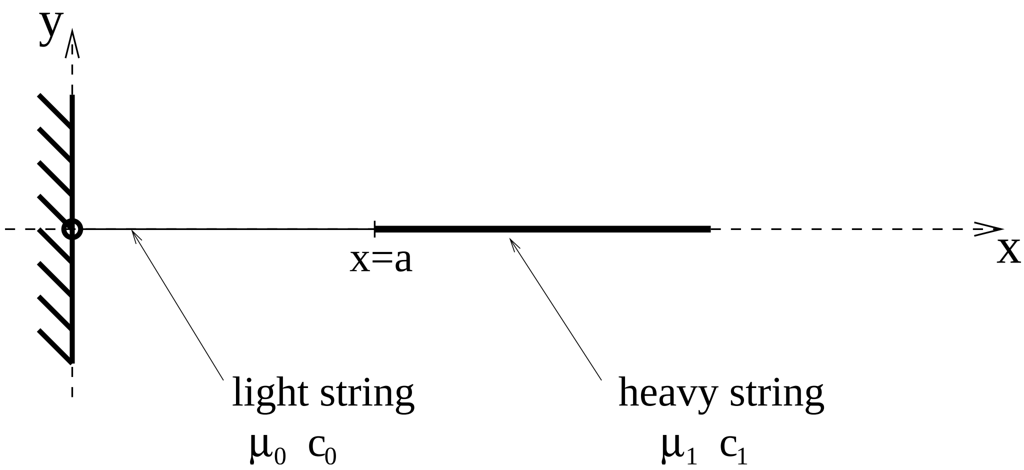

Figure 2: Transmission and reflection in strings.

4.2 Sums over Histories

The second way is to use the "sum over histories" idea developed in the lecture notes. The reflection and transmission amplitudes at the interface of two *half-infinite* strings can be obtained by analogy with sound waves. For example, for waves going from the light string to the heavy string we have

$$\begin{aligned} R_{01} &= \frac{z_0 - z_1}{z_0 + z_1}; \\ T_{01} &= \frac{2z_0}{z_0 + z_1}, \end{aligned} \tag{3}$$

where the impedance for strings is $z = \mu c$.

(a) What are the transmission and reflection coefficients for waves going from the heavy string to the light string, R_{10} and T_{10}?

(b) What is the complex amplitude (at $x = a$) of the wave that was reflected from the interface between the strings *without* entering the light string? What about the wave that passed into the light string, was reflected from the boundary, and then passed back into the heavy string?

(c) *Challenge problem!* Generalizing your result in part (b), obtain the full complex amplitude of the right-moving wave in the heavy string at $x = a$ by summing over all possible back-and-forth reflections within the light part of the string. Use this result to find r.
HINT: To simplify your expression, use the formula for the sum of a geometric series (which also works for complex numbers): $\underline{a} + \underline{a}\,\underline{x} + \underline{a}\,\underline{x}^2 + ... = \underline{a}/(1 - \underline{x})$.

5 Using Interference to Study Stars

Two telescopes, placed at a distance D from one another, receive light of wavelength λ from a distant star. The star is at an angle θ above horizon. Light from both telescopes is then redirected to the sensor S which is located *exactly halfway* between them (see Fig. 3).

(a) What is the difference in the distance traveled by the light received by telescope A and telescope B? Express your answer in terms of the angle θ and the distance D. (*Hint*: The distance from the star to Earth is **much** greater than D.)

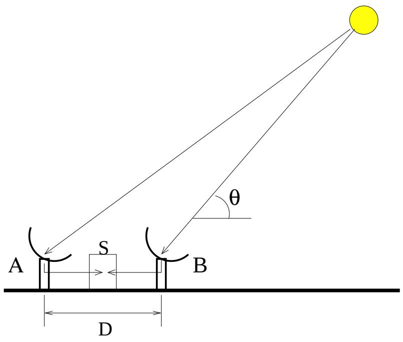

Figure 3: Stellar Interference.

(b) What is the intensity of light measured by the sensor? Express your answer in terms of the angle θ, distance D, wavelength λ, and the intensity I_0 that would be measured if only one of the telescopes were used.

(c) What are the values of $\cos\theta$ for which the intensity is at a maximum? At a minimum?

(d) If the star is moving, θ will change with time. By how much does $\cos\theta$ have to change to cause a large variation in the light intensity (e.g., from a maximum to the neighboring minimum)?

(e) In a typical situation, $D \gg \lambda$ (typical numbers would be $D \approx 1$ m, $\lambda \approx 10^{-6}$ m). Explain how the setup described in this problem can be used to detect tiny shifts in star positions, unobservable by naked eye.