# Physics 214 Homework Assignment Eight Solutions

October 29, 2004

## 1 Michelson Interferometer with Microwaves

a) If we already register a maximum intensity with our interferometer and then move Reflector (1) along Arm (1) in order to find another maximum, we must move the reflector a distance of $\lambda/2$. This is because a change in reflector position of $\Delta X$ will result in a total path length change of $2\Delta X$, and we want to change this path length by one wavelength to find a successive maximum. Given the data: $X = 20.0$ cm, 23.5 cm, 26.5 cm, 30.0 cm and 33.4 cm, we find that the average $\Delta X$ is $\Delta X_{ave} = 3.35$ cm. Therefore the wavelength is

$$\lambda = 2\Delta X_{ave} = 6.7\ cm \tag{1}$$

b) In a space of a thickness $l$, we would normally be able to fit $l/\lambda$ wavelengths of microwave. If we then place a sheet of polyethylene with index of refraction $n$ and thickness $l$ in that space, the wavelength becomes shorter by a factor of $n$, and thus we'd be able to fit $nl/\lambda$ wavelengths in the same space (see Figure 1 on last page). This gives us an overall shift by $l(n-1)/\lambda$ wavelengths for each pass through the polyethylene. The phase difference introduced will be $2\pi$ for each extra wavelength and in our interferometer, the microwaves will pass through this sheet twice. Therefore we will get a total phase difference of

$$\Delta\phi = \frac{4\pi l}{\lambda}(n-1) = \frac{4\pi(0.60)}{6.7}(1.6-1) \approx 0.68\ rad \tag{2}$$

c) Since you will get an extra bit of wavelength from the addition of the polyethylene, you will want to shorten the path length that the wave will travel to compensate. This can be done by moving Reflector (1) towards the partial reflector at the center.

d) Adding the polyethylene will cause an increase in the optical path length by $l(n-1)$ for each pass through the medium. The microwaves will pass through it twice. We will need to compensate by moving the reflector towards the center by $\Delta X$ such that this change increases the path length by $2\Delta X$. Therefore, in order to reestablish zero phase difference

$$2\Delta X = 2l(n-1) \tag{3}$$

and therefore

$$\Delta X = l(n-1) = (0.60\ cm)(1.6-1) = 0.36\ cm \tag{4}$$

## 2 Impedance Matching

a) For a sinusoidal wave traveling from the left and moving to the right, with a massless slip ring attached to a dashpot at $x = 0$, we have the solution

$$y_0(x \le 0, t) = \Re\left[\underline{A}e^{-i\omega t}\left(e^{ikx} + \underline{R}e^{-ikx}\right)\right] \tag{5}$$

The first term in the above equation corresponds to the incoming wave which is moving in the positive x-direction. The second term corresponds to the wave which is reflected back in the negative x-direction by the ring-dashpot system.

b) The ring has two forces acting on it: the tension force from the string to its left and the drag force from the dashpot. These are

$$F_{string} = -\tau \frac{\partial y(0,t)}{\partial x} \tag{6}$$

and

$$F_{drag} = -bv_y(0,t) = -b\frac{\partial y(0,t)}{\partial t} \quad . \tag{7}$$

Newton's force law tells us that $\sum \overrightarrow{F} = m\overrightarrow{a}$, and since $m = 0$ for the ring

$$\sum \overrightarrow{F} = -\tau \frac{\partial y(0,t)}{\partial x} - b\frac{\partial y(0,t)}{\partial t} = 0 \tag{8}$$

c) Now we can simply plug our solution into the above boundary condition. The partial derivatives will be

$$\frac{\partial y(0,t)}{\partial x} = \Re\left[\underline{A}e^{-i\omega t}\left(ik - ik\underline{R}\right)\right] \tag{9}$$

and

$$\frac{\partial y(0,t)}{\partial t} = \Re\left[-i\omega\underline{A}e^{-i\omega t}\left(1 + \underline{R}\right)\right], \tag{10}$$

and these give us the boundary condition

$$\Re\left[-ib\omega\underline{A}e^{-i\omega t}\left(1+\underline{R}\right) + ik\tau\underline{A}e^{-i\omega t}\left(1-\underline{R}\right)\right] = 0. \tag{11}$$

Factoring out the $i\underline{A}e^{-i\omega t}$, we find that this expression is only zero for all times if

$$-b\omega\left(1+\underline{R}\right) + k\tau\left(1-\underline{R}\right) = 0 \tag{12}$$

and, solving for $\underline{R}$, this gives us

$$\underline{R} = \frac{k\tau - b\omega}{k\tau + b\omega} = \frac{z\omega - b\omega}{z\omega + b\omega} = \frac{z-b}{z+b} \tag{13}$$

d) Using our result for the reflected wave's complex amplitude

$$\underline{R} = \frac{z-b}{z+b} \tag{14}$$

we see that when $b = z$, $\underline{R} = 0$ and thus we get no reflected wave. In other words, we get no reflection if we impose "impedance matching" by making the damping coefficient match the impedence of the string: $b = z$. In this situation, all of the energy in the wave is dissipated by friction in the dashpot. This is one example of impedance matching, where we match the impedances in order to get zero reflection from an interface.

# 3 Reflection and Transmission with an Attached Dashpot

a) Now if we change the situation in problem (2) by tying string to both sides of the slip ring, we get the following solution

$$y_0(x \le 0, t) = \Re\left[\underline{A}e^{-i\omega t}\left(e^{ikx} + \underline{R}e^{-ikx}\right)\right] \tag{15}$$

$$y_1(x \geq 0, t) = \Re\left[\underline{A}e^{-i\omega t}\left(\underline{T}e^{ikx}\right)\right] \tag{16}$$

The first term of $y_0$ is the incident wave and the second term is the reflected wave; $y_1$ is the transmitted wave, which will travel past the slip ring and continue on in the positive x-direction.

b) At the point $x = 0$, we must have

$$y_0(0, t) = y_1(0, t) \tag{17}$$

because the solutions must match at the ring due to the continuity of the string through this point, as in the lecture notes.

c) Now the ring has three forces acting on it: the tension force from the string on the left, the tension force from the string on the right and the drag force from the dashpot. These are, respectively,

$$F_{left} = -\tau \frac{\partial y_0(0, t)}{\partial x}, \tag{18}$$

$$F_{right} = \tau \frac{\partial y_1(0, t)}{\partial x} \tag{19}$$

and

$$F_{drag} = -bv_y = -b \frac{\partial y_1(0, t)}{\partial t} \tag{20}$$

where we could have similarly used $y_0$ to calculate the velocity used in the drag force equation. The sum of the forces will be equal to zero since the ring is massless and hence

$$\tau \left( \frac{\partial y_1(0, t)}{\partial x} - \frac{\partial y_0(0, t)}{\partial x} \right) - b \frac{\partial y_1(0, t)}{\partial t} = 0. \tag{21}$$

d) Using the boundary condition from part (b) and plugging in our solution tells us that

$$\Re\left[\underline{A}e^{-i\omega t}\left(1 + \underline{R}\right)\right] = \Re\left[\underline{A}e^{-i\omega t}\underline{T}\right] \tag{22}$$

and we can bring everything into one $\Re[\,]$ and factor out the $\underline{A}e^{-i\omega t}$ to find

$$1 + \underline{R} = \underline{T}. \tag{23}$$

Plugging our solution into the second boundary condition (our differential equation from part (c)) tells us that

$$\Re\left[\tau\underline{A}e^{-i\omega t}ik\underline{T}\right] - \Re\left[\tau\underline{A}e^{-i\omega t}ik\left(1 - \underline{R}\right)\right] - \Re\left[-i\omega b\underline{A}e^{-i\omega t}\underline{T}\right] = 0 \tag{24}$$

and this simplifies to give

$$k\tau\underline{T} - k\tau\left(1 - \underline{R}\right) + \omega b\underline{T} = 0. \tag{25}$$

We now have two equations and two unknowns. Solving for the unknowns, we find that

$$\underline{R} = -\frac{b}{2z + b} \tag{26}$$

and

$$\underline{T} = \frac{2z}{2z + b} \tag{27}$$

e) As $b \to 0$, $\underline{R} \to 0$ and $\underline{T} \to 1$. The system acts as a regular continuous string with no external force applied at $x = 0$, so there is no reflection and the wave travels on with undiminished amplitude.

f) As $b \to \infty 0$, $\underline{R} \to -1$ and $\underline{T} \to 0$. As the damping coefficient gets very large, it will not allow any motion at all, so $x = 0$ approaches the behavior of a fixed end. The reflected wave is inverted and there is no transmitted wave.

# 4 Transmission and Reflection in Strings (II)

## 4.1 Matching at the Boundary

a) Using the boundary condition at $x = 0$, we know that

$$y_0(0,t) = 0 \tag{28}$$

and using our solution, this is tells us that

$$\Re\left[e^{-i\omega t}\left(\underline{B} + \underline{C}\right)\right] = 0 \tag{29}$$

and therefore

$$\underline{B} = -\underline{C}. \tag{30}$$

Then the solution for $y_0$ simplifies to

$$y_0(x < a, t) = \Re\left[e^{-i\omega t}\underline{B}\left(e^{-ik_0 x} - e^{ik_0 x}\right)\right]. \tag{31}$$

Using the fact that

$$e^{-ik_0 x} - e^{ik_0 x} = -2i\sin(k_0 x) \tag{32}$$

and defining

$$\underline{D} = -2i\underline{B} \tag{33}$$

we get

$$y_0(x < a, t) = \Re\left[e^{-i\omega t}\underline{D}\sin(k_0 x)\right]. \tag{34}$$

b) At the point $x = a$, the string is continuous and without any breaks. Therefore the two solutions should match each other at that point

$$y_0(a,t) = y_1(a,t). \tag{35}$$

We must also have force balance at the point $x = a$, which means that the total y-component of the tension should be zero at the boundary

$$\tau\frac{\partial y_0(a,t)}{\partial x} = \tau\frac{\partial y_1(a,t)}{\partial x}. \tag{36}$$

c) Plugging our solution into the first boundary condition tells us that

$$\Re\left[e^{-i\omega t}\underline{D}\sin(k_0 a)\right] = \Re\left[\underline{A}e^{-i\omega t}\left(1 + \underline{r}\right)\right] \tag{37}$$

and hence

$$\underline{D}\sin(k_0 a) = \underline{A}\left(1 + \underline{r}\right). \tag{38}$$

The second boundary condition tells us that

$$\Re\left[e^{-i\omega t}\underline{D}k_0\cos(k_0 a)\right] = \Re\left[\underline{A}e^{-i\omega t}ik_1\left(\underline{r} - 1\right)\right] \tag{39}$$

and hence

$$k_0\underline{D}\cos(k_0 a) = ik_1\underline{A}\left(\underline{r} - 1\right). \tag{40}$$

We can use these two conditions to solve for $\underline{\mathrm{r}}$. Dividing the first equation by the second gives us

$$\frac{1}{k_0}\tan(k_0 a) = \frac{1 + \underline{r}}{ik_1(\underline{r} - 1)} \tag{41}$$

and we can then solve for $\underline{r}$

$$\underline{r} = \frac{ik_1\tan(k_0 a) + k_0}{ik_1\tan(k_0 a) - k_0} = \frac{ic_0\tan(c_1 k_1 a/c_0) + c_1}{ic_0\tan(c_1 k_1 a/c_0) - c_1} \tag{42}$$

where in the last step we used that fact that $c_0 k_0 = c_1 k_1 = \omega$. Note that if $c_0 = c_1$, this expression reduces to $\underline{r} = -e^{2ik_0 a}$, which is correct: with no discontinuity at $x = a$, we just have a factor of $-1$ for reflection from the fixed end and a propagation (phase) factor of $e^{2ik_0 a}$.

## 4.2 Sums over Histories

a) The transmission and reflection coefficients for waves going from the light string to the heavy string are

$$R_{01} = \frac{z_0 - z_1}{z_0 + z_1} \tag{43}$$

and

$$T_{01} = \frac{2z_0}{z_0 + z_1}; \tag{44}$$

similarly, the transmission and reflection coefficients for waves going from the heavy string to the light string are

$$R_{10} = \frac{z_1 - z_0}{z_1 + z_0} \tag{45}$$

and

$$T_{10} = \frac{2z_1}{z_1 + z_0} \tag{46}$$

where all we've done is switch the ones and zeroes.

b) A part of the incident wave is going to reflect back out when it hits the heavy-light boundary. The complex amplitude of this, which we will call $\underline{A_0}$ is just the incident amplitude multiplied by the appropriate reflection coefficient

$$\underline{A_0} = R_{10}\underline{A}. \tag{47}$$

Another part of the incident wave will be transmitted through the heavy-light boundary, reflect off of the fixed boundary, and transmit back to the heavy string. The complex amplitude $\underline{A_1}$ of this part will be the incident amplitude times the two appropriate transmission coefficients and multiplied by its phase shift from traveling the extra distance $2a$, with a minus sign for the reflection from the fixed end

$$\underline{A_1} = -T_{10}p^2T_{01}\underline{A} \tag{48}$$

where we have defined the phase shift factor as in lecture: $p \equiv e^{ik_0a}$.

c) If sum over all possible "histories" to get the total reflected amplitude $\underline{A_{ref}}$, we get the infinite series

$$\underline{A_{ref}} = \left(R_{10} - T_{10}p^2T_{01} + T_{10}R_{01}p^4T_{01} - T_{10}p^6R_{01}^2T_{01} + ...\right)\underline{A} \tag{49}$$

and this equals

$$\underline{A_{ref}} = \left[R_{10} - T_{10}T_{01}p^2\sum_{n=0}^{\infty}\left(-R_{01}p^2\right)^n\right]\underline{A} \tag{50}$$

which, using the geometric series formula, is also

$$\underline{A_{ref}} = \underline{A}\left[R_{10} - \frac{T_{10}T_{01}p^2}{1 + R_{01}p^2}\right]. \tag{51}$$

So the reflection coefficient is

$$\underline{r} = \frac{\underline{A_{ref}}}{\underline{A}} = R_{10} - \frac{T_{10}T_{01}p^2}{1 + R_{01}p^2}. \tag{52}$$

Now we can do some things to get back our answer from part (4.1):

1) plug in what we know for the reflection and transmission coefficients from part (a),
2) use Euler's formula to expand $p$: $p = e^{ik_0a} = \cos(k_0a) + i\sin(k_0a)$
3) and do a lot of algebra.

At the end of the day, we get back our answer from part (4.1)

$$\underline{r} = \frac{ic_0\tan(c_1k_1a/c_0) + c_1}{ic_0\tan(c_1k_1a/c_0) - c_1} \tag{53}$$

## 5 Using Interference to Study Stars

a) Since we know that the star is is far away compared to the distance between the telescopes, we can assume that the two light paths are essentially parallel (see Figure 2 on last page). From the diagram, we see that the path difference is equal to $D\cos\theta$.

b) The signal coming to each telescope is coming from the same source, so the only reason why the measured intensity differs is because of the phase difference due to the path length difference calculated in part (a). Each signal by itself would have an amplitude $\sqrt{I_0}$. If we say that the signal at B is $\Re\left[\sqrt{I_0}e^{i\omega t}\right]$, then the signal at A is $\Re\left[\sqrt{I_0}e^{i(\omega t+\Delta\phi(\theta))}\right]$, where it is just off by the phase difference caused by the path difference:

$$\Delta\phi(\theta) = 2\pi\frac{D\cos\theta}{\lambda}. \tag{54}$$

The combined signal is simply the superposition, or sum, of the two signals and the intensity $I(\theta)$ is the squared magnitude of this:

$$I(\theta) = |\sqrt{I_0}e^{i\omega t}\left(1+e^{i\Delta\phi(\theta)}\right)|^2 = 2I_0\left(1+\cos\left(\frac{2\pi D}{\lambda}\cos\theta\right)\right). \tag{55}$$

Using the half-angle trig identities, this becomes

$$I(\theta) = 4I_0\cos^2\left(\frac{\pi D}{\lambda}\cos(\theta)\right). \tag{56}$$

c) The intensity calculated above will be a minimum when $\cos(\frac{\pi D}{\lambda}\cos(\theta)) = 0$. This happens when

$$\cos(\theta) = \left(n+\frac{1}{2}\right)\frac{\lambda}{D} \tag{57}$$

where $n$ is an integer, positive or negative. Likewise, the intensity becomes a maximum when $\cos(\frac{\pi D}{\lambda}\cos(\theta)) = \pm 1$, and this happens when

$$\cos(\theta) = \frac{n\lambda}{D} \tag{58}$$

where $n$ is again any integer.

d) As we can see from part (c), in order to change the intensity from a maximum to a neighboring minimum, we only have to change $\cos\theta$ by $\lambda/2D$.

e) So if we say that

$$D \approx 1\ m \tag{59}$$

and

$$\lambda \approx 10^{-6}\ m \tag{60}$$

then

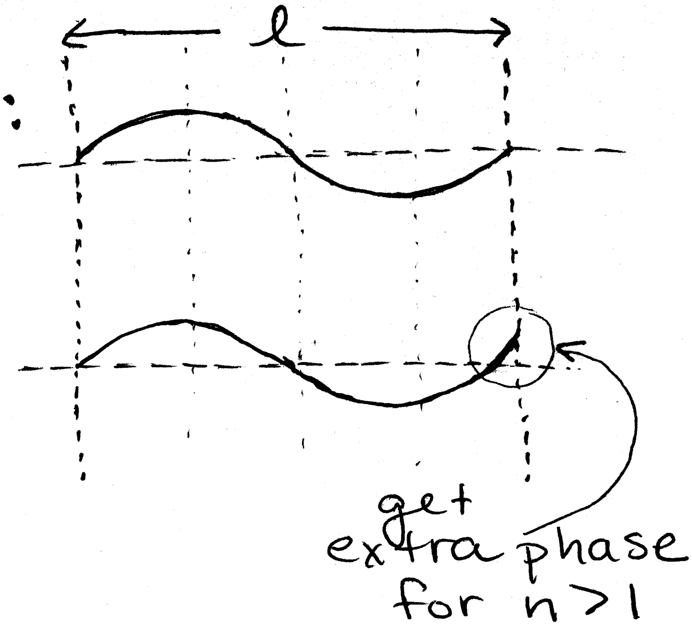
$$\frac{\lambda}{2D} \approx 5\times 10^{-7} \tag{61}$$

and thus it is very easy to measure a shift is $\cos\theta$ of $5\times 10^{-7}$. If $\cos\theta \approx 1$, then this corresponds to a shift of $\Delta\theta \approx 0.001$ rad, which is too tiny to detect with the naked eye.

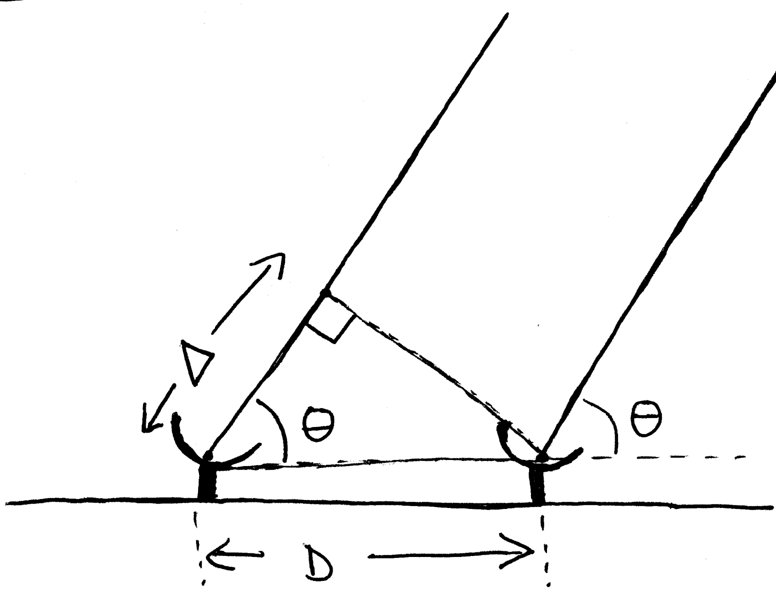
## Figure 1

Without polyethylene:
(n=1)

With polyethylene:
(n>1)




## Figure 2




$$\Delta = D\cos\theta$$