Physics 214 - Problem Set 9 - Solutions

N-narrow slit interference:

$$I(\theta) = I_0 \frac{\sin^2(Nkd\sin(\theta)/2)}{\sin^2(kd\sin(\theta)/2)} \tag{1}$$

N-finite slit interference:

$$I(\theta) = I_0 \frac{\sin^2(ka\sin(\theta)/2)}{(ka\sin(\theta)/2)^2} \frac{\sin^2(Nkd\sin(\theta)/2)}{\sin^2(kd\sin(\theta)/2)} \tag{2}$$

If a screen is a distance L from the source, and measurements are made at position x (relative to the maximum intensity on the screen), the relation between the angle and x is:

$$\theta = \tan^{-1}\frac{x}{L} \tag{3}$$

1 N-slit interference and diffraction

Figure 1 in the problem set plots the intensity of an N-finite slit interference pattern (see (2)). Since the screen is 4.0 m from the slits, $x \in (-0.02 \text{ m}, 0.02 \text{ m})$, and $.02 \text{ m}/4.0 \text{ m} \ll 1$ the small angle approximation says

$$\sin\theta(x) = \sin\tan^{-1}\frac{x}{4.0 \text{ m}} \approx \frac{x}{4.0 \text{ m}}. \tag{4}$$

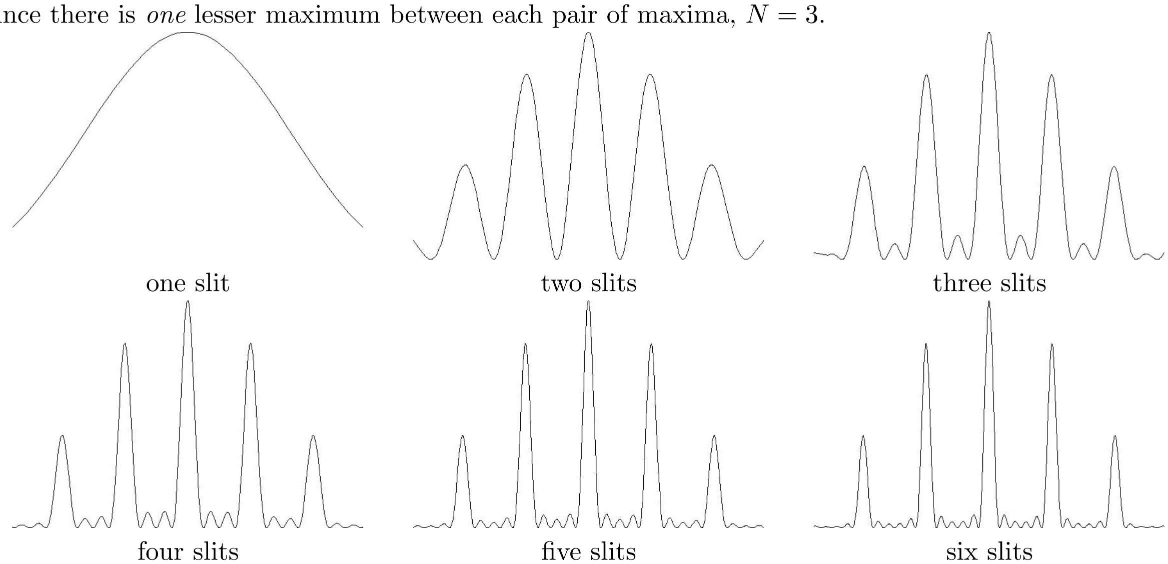
Substituting (4) into (2) gives

$$I(x) = I_0 \frac{\sin^2(ka\frac{x}{8.0 \text{ m}})}{(ka\frac{x}{8.0 \text{ m}})^2} \frac{\sin^2(Nkd\frac{x}{8.0 \text{ m}})}{\sin^2(kd\frac{x}{8.0 \text{ m}})} \tag{5}$$

The wavelength, ($\lambda = 720$ nm), determines the wave vector k.

$$k = 2\pi/\lambda = 8.73 \times 10^6 \text{ m}^{-1} \tag{6}$$

a) Since there is *one* lesser maximum between each pair of maxima, $N = 3$.



b) The second primary maximum occurs when $\sin(kd\frac{x}{8 \text{ m}}) = 0$. Reading off the graph: $x_2 = x_{2nd\ max} = 0.0072$ m. This occurs when $kd\frac{x_2}{8 \text{ m}} = \pi$, or:

$$d = \frac{8\text{ m} \times \pi}{x_2 k} = \frac{8\text{ m} \times 3.14}{0.0072\text{ m} \times (8.73 \times 10^6\text{ m}^{-1})} = 0.00040\text{ m} = 4.0 \times 10^{-4}\text{ m} \tag{7}$$

Notice how this is the same magnitude as the width of a human hair (0.0001 m).
c) Recall that for $b = 0$ and $b = x_2$

$$\lim_{x \to b} \frac{\sin^2(Nkd\frac{x}{8m})}{\sin^2(kd\frac{x}{8m})} = N^2 = 9 \tag{8}$$

and

$$\lim_{x \to 0} \frac{\sin^2(ka\frac{x}{8m})}{(ka\frac{x}{8m})^2} = 1. \tag{9}$$

Therefore

$$\frac{I(x_2)}{I(0)} = \frac{I_0 \times \frac{\sin^2(ka\frac{x_2}{8m})}{(ka\frac{x_2}{8m})^2} \times 9}{I_0 \times 1 \times 9} = \frac{\sin^2(ka\frac{x_2}{8m})}{(ka\frac{x_2}{8m})^2} \tag{10}$$

The ratio comes from the graph

$$\frac{I(x_2)}{I(0)} = \frac{0.115}{0.18} = 0.64 \tag{11}$$

The slit width a is found by solving

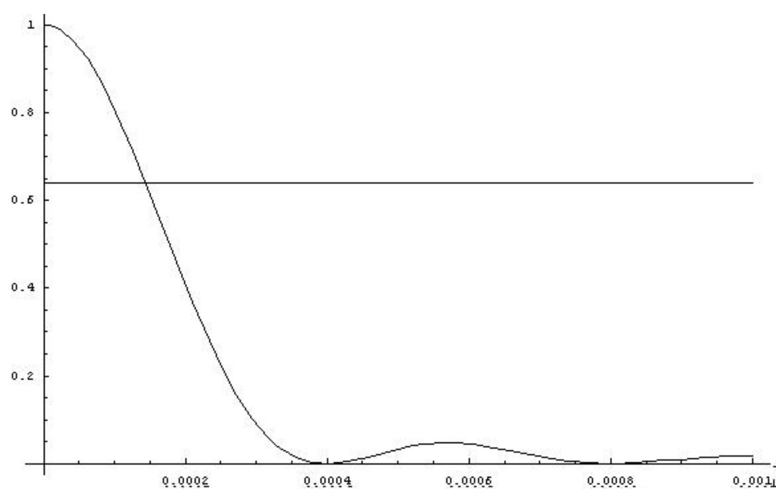
$$0.64 = \frac{\sin^2(ka\frac{x_2}{8\text{ m}})}{(ka\frac{x_2}{8\text{ m}})^2} \tag{12}$$

The easiest way to solve this problem numerically is to plot $y_1(a)$ and $y_2(a)$ defined by

$$y_1(a) = 0.64 \tag{13}$$

$$y_2(a) = \frac{\sin^2(ka\frac{x_2}{8\text{ m}})}{(ka\frac{x_2}{8\text{ m}})^2} = \frac{\sin^2(8.73 \times 10^6\text{ m}^{-1} \times a\frac{0.0072\text{ m}}{8\text{ m}})}{(8.73 \times 10^6\text{ m}^{-1} \times a\frac{0.0072\text{ m}}{8\text{ m}})^2} \tag{14}$$

The intersection point[1] is approximately $a = 0.000145\text{ m} = 1.45 \times 10^{-4}$ m.



y_1, y_2 vs. a [m]

[1]On the TI-83 the command is '2nd' 'Trace' '5-Intersect'

d) The intensity for one slit at $x = 0$ is (setting $N = 1$ and $\theta = 0$)

$$I = \lim_{\sin(\theta)\to 0} I_0 \frac{\sin^2(ka\sin(\theta)/2)}{(ka\sin(\theta)/2)^2} \frac{\sin^2(1\times kd\sin(\theta)/2)}{\sin^2(kd\sin(\theta)/2)} \tag{15}$$

$$= \lim_{\sin(\theta)\to 0} I_0 \frac{\sin^2(ka\sin(\theta)/2)}{(ka\sin(\theta)/2)^2} \tag{16}$$

$$= \lim_{\sin(\theta)\to 0} I_0 \frac{(ka\sin(\theta)/2)^2}{(ka\sin(\theta)/2)^2} \tag{17}$$

$$= I_0 \tag{18}$$

The intensity for three slits at $x = 0$ is (setting $N = 3$ and $\theta = 0$)

$$I = \lim_{\sin(\theta)\to 0} I_0 \frac{\sin^2(ka\sin(\theta)/2)}{(ka\sin(\theta)/2)^2} \frac{\sin^2(3\times kd\sin(\theta)/2)}{\sin^2(kd\sin(\theta)/2)} \tag{19}$$

$$= \lim_{\sin(\theta)\to 0} I_0 \frac{\sin^2(ka\sin(\theta)/2)}{(ka\sin(\theta)/2)^2} \times 9 \tag{20}$$

$$= \lim_{\sin(\theta)\to 0} I_0 \frac{(ka\sin(\theta)/2)^2}{(ka\sin(\theta)/2)^2} \times 9 \tag{21}$$

$$= 9I_0 \tag{22}$$

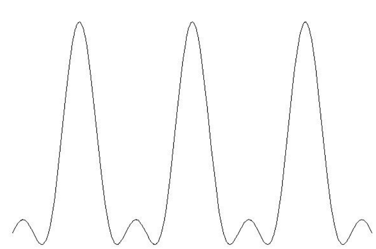
From the graph, it is apparent that $9I_0 = 0.18\ mW^2/m^2$ which implies that $I_0 = 0.02\ mW^2/m^2$ is the intensity of one slit at $x = 0$.
e) Assume $a \ll \lambda$. Since $k = 2\pi/\lambda$, $ak = 2\pi\frac{a}{\lambda} \ll 1$ and

$$I(x) = I_0 \frac{\sin^2(ka\frac{x}{8.0\text{ m}})}{(ka\frac{x}{8.0\text{ m}})^2} \frac{\sin^2(Nkd\frac{x}{8.0\text{ m}})}{\sin^2(kd\frac{x}{8.0\text{ m}})} \tag{23}$$

$$= I_0 \frac{(ka\frac{x}{8.0\text{ m}})^2}{(ka\frac{x}{8.0\text{ m}})^2} \frac{\sin^2(Nkd\frac{x}{8.0\text{ m}})}{\sin^2(kd\frac{x}{8.0\text{ m}})} \tag{24}$$

$$= I_0 \frac{\sin^2(Nkd\frac{x}{8.0\text{ m}})}{\sin^2(kd\frac{x}{8.0\text{ m}})} \tag{25}$$

The slits become narrow.



the interference pattern when slits are narrow

2 Measuring wavelengths with a grating

a) A grating gives rise to an N-narrow slit interference pattern where $N \gg 1$. Since N is large, the non-primary maxima are negligible. The primary (and only) maxima occur when

$$d \sin(\theta) = n\lambda \quad n = 0, 1, 2, 3, \ldots \tag{26}$$

The table below shows that there are only two wavelengths (approximately 449.3 nm and 651.1 nm). The first column is data that is given. The second column is just $d \sin(\theta)$. The third and fourth columns are found by *inspection*.

$$d = \frac{1 \text{ cm}}{5000} = 0.000002 \text{ m} \tag{27}$$

θ [°]	$d\sin(\theta)$[nm]	=	n	×	λ [nm]
12.98	449.2	=	1	×	449.2
19.0	651	=	1	×	651.1
26.7	899	=	2	×	449.3
40.6	1301	=	2	×	650.8
42.4	1349	=	3	×	449.5
63.9	1796	=	4	×	449.0
77.6	1953	=	3	×	651.1

b) Now suppose $d = 1 \text{ cm}/2000 = 5000$ nm. Given n and λ, the angle θ is

$$\theta_n = \sin^{-1} \frac{n\lambda}{d} \tag{28}$$

The table below shows the computations. When n is "large", the inverse sin breaks down and there are no more visible light rays. The total number of unique angles ($\neq 0$) is $18 = 11 + 7$.

n	$\sin^{-1}\left(\frac{449.3 \text{ nm} \times n}{5000 \text{ nm}}\right)$	$\left(\sin^{-1} \frac{651.1 \text{ nm} \times n}{5000 \text{ nm}}\right)$
1	5.2	7.5
2	10.4	15.1
3	15.6	23.0
4	21.1	31.4
5	26.7	40.6
6	32.6	51.4
7	39.0	65.7
8	46.0	×
9	54.0	×
10	64.0	×
11	81.3	×
12	×	×

3 Resolving Power of a Grating

A grating with N slits gives rise to the following interference pattern.

$$I(\theta) = I_0 \frac{\sin^2(Nkd\sin(\theta)/2)}{\sin^2(kd\sin(\theta)/2)} \tag{29}$$

a) The primary maxima occur at angles θ_m given by

$$kd\sin(\theta_m)/2 = m\pi \quad m = 0, 1, 2, 3, \ldots \tag{30}$$

or

$$\theta_m = \sin^{-1}\left(\frac{2m\pi}{kd}\right) = \sin^{-1}\left(\frac{m\lambda}{d}\right) \tag{31}$$

If there are two wavelengths λ and $\lambda' = \lambda + \Delta\lambda$ then

$$\delta_m = \theta'_m - \theta_m \tag{32}$$
$$= \sin^{-1}\left(\frac{m\lambda'}{d}\right) - \sin^{-1}\left(\frac{m\lambda}{d}\right) \tag{33}$$
$$= \sin^{-1}\left(\frac{m\lambda}{d} + \frac{m\Delta\lambda}{d}\right) - \sin^{-1}\left(\frac{m\lambda}{d}\right) \tag{34}$$
$$\tag{35}$$

If $\Delta\lambda$ is small, a first order taylor expansion is sufficient.

$$\sin^{-1}(x+\epsilon) \approx \sin^{-1} x + \frac{\epsilon}{\sqrt{1-x^2}} \tag{36}$$

$$\delta_m = \left[\sin^{-1}\left(\frac{m\lambda}{d}\right) + \frac{\frac{m\Delta\lambda}{d}}{\sqrt{1-\left(\frac{m\lambda}{d}\right)^2}}\right] - \sin^{-1}\left(\frac{m\lambda}{d}\right) \tag{37}$$
$$= \frac{\frac{m\Delta\lambda}{d}}{\sqrt{1-\left(\frac{m\lambda}{d}\right)^2}} \tag{38}$$
$$= \frac{m\Delta\lambda}{\sqrt{d^2 - m^2\lambda^2}} \tag{39}$$

b) The m'th principla maxima occurs when

$$kd\sin(\theta)/2 = m\pi. \tag{40}$$

Multiplying by N gives the parameter to $\sin^2$ in the numerator

$$Nkd\sin(\theta_m)/2 = Nm\pi. \tag{41}$$

The next minimum will occur after one more π is added to the right hand side.

$$Nkd\sin(\theta_m + \epsilon_m)/2 = (Nm+1)\pi \tag{42}$$

Solving for ϵ_m gives

$$\epsilon_m = \sin^{-1}\left(\frac{\left(m + \frac{1}{N}\right)\lambda}{d}\right) - \theta_m \tag{43}$$
$$= \sin^{-1}\left(\frac{m\lambda}{d} + \frac{\lambda}{Nd}\right) - \sin^{-1}\left(\frac{m\lambda}{d}\right) \tag{44}$$
$$\tag{45}$$

Assuming $N \gg 1$ and invoking (36) gives

$$\epsilon_m = \frac{\frac{\lambda}{Nd}}{\sqrt{1-\left(\frac{m\lambda}{d}\right)^2}} \tag{46}$$
$$= \frac{\lambda}{N\sqrt{d^2 - m^2\lambda^2}} \tag{47}$$

c+d) The grating can distinguish two wavelengths if the separation is greater than the width (otherwise the two signals will overlap)

$$\delta_m > \epsilon_m \tag{48}$$

$$\frac{m\Delta\lambda}{\sqrt{d^2 - m^2\lambda^2}} > \frac{\lambda}{N\sqrt{d^2 - m^2\lambda^2}} \tag{49}$$

$$\Delta\lambda > \frac{\lambda}{Nm} \tag{50}$$

4 Reflection Grating

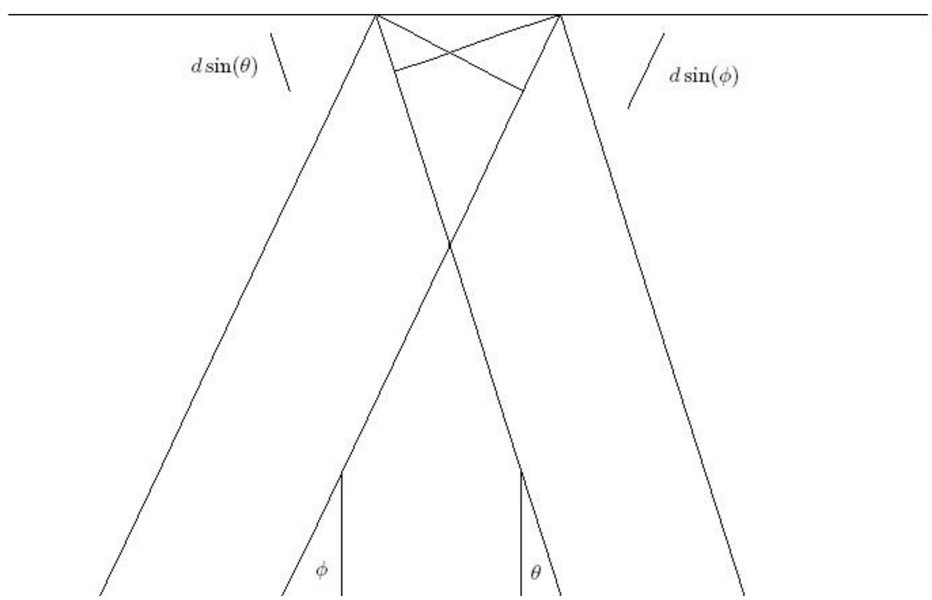
a) The principle maxima occur at angles θ_m where

$$d\sin(\theta_m) = m\lambda \;\;\Rightarrow\;\; \theta_m = \sin^{-1}\left(\frac{m\lambda}{d}\right) \tag{51}$$

b) The answer does not depend on N. When $N \to \infty$ the non-primary maxima are negligible and the primary maxima are very narrow. This is why gratings are useful; gratings can restrict light to specific angles. (see figures in question one)
c) One way to rewrite (51) with $m = 1$ is

$$\lambda(\theta_1) = d\sin(\theta_1). \tag{52}$$

In other words, every angle is the first maximum for some wavelength, changing the angle changes the wavelength. A light source whose wavelength is a continuous function of angle is usually called a rainbow!
d)



Constructive interference occurs when the relative phase differences add to a multiple of the wavelength.

$$d\sin(\phi) - d\sin(\theta_m) = m\lambda \;\; m = 0, \pm1, \pm2, \pm3, ... \tag{53}$$

Notice how m can be negative. The solution for θ_m is

$$\theta_m = \sin^{-1}\left(-\frac{m\lambda}{d} + \sin(\phi)\right) \tag{54}$$

e) Let $\phi = \frac{\pi}{2} - \alpha$, $\theta = \frac{\pi}{2} - \beta$, and assume $\theta_m > \phi$.

$$\sin\theta_m = -\frac{m\lambda}{d} + \sin(\phi) \tag{55}$$

$$\sin\left(\frac{\pi}{2} - \beta_m\right) = -\frac{m\lambda}{d} + \sin\left(\frac{\pi}{2} - \alpha\right) \tag{56}$$

To second order:

$$\sin\left(\frac{\pi}{2} - \beta_m\right) \approx 1 - \frac{1}{2}\beta_m^2 \tag{57}$$

and

$$\sin\left(\frac{\pi}{2} - \alpha\right) \approx 1 - \frac{1}{2}\alpha^2 \tag{58}$$

Substituting (57) and (58) into (56) gives

$$1 - \frac{1}{2}\beta_m^2 = -\frac{m\lambda}{d} + 1 - \frac{1}{2}\alpha^2 \tag{59}$$

$$\beta_m^2 = \alpha^2 + \frac{2m\lambda}{d} \tag{60}$$

Taking the square root and using the approximation $\sqrt{a^2+\epsilon} \approx a + \epsilon/(2a)$ (assuming the second term is small) gives

$$\beta_m = \alpha + \frac{m\lambda}{d\alpha} \tag{61}$$

f) Suppose $d \gg \lambda$, then the above approximation is valid and

$$\beta_{m+1} - \beta_m = \left(\alpha + \frac{(m+1)\lambda}{d\alpha}\right) - \left(\alpha + \frac{(m)\lambda}{d\alpha}\right) \tag{62}$$

$$= \frac{\lambda}{d\alpha}. \tag{63}$$

This set up is useful because small α gives a large separation.

5 Checking some interesting limiting cases

a)

$$I_a(\theta) = I_0 \frac{\sin^2(ka\sin(\theta)/2)}{(ka\sin(\theta)/2)^2} \tag{64}$$

When $a \to 0$, $\sin^2(\epsilon) \approx \epsilon^2$ and

$$\lim_{a\to 0} I_0 \frac{\sin^2(ka\sin(\theta)/2)}{(ka\sin(\theta)/2)^2} = I_0 \frac{(ka\sin(\theta)/2)^2}{(ka\sin(\theta)/2)^2} = I_0 \tag{65}$$

The intensity is independent of angle (Huygens' Principle)
b)

$$I = I_1 \frac{\sin^2(ka\sin(\theta)/2)}{(ka\sin(\theta)/2)^2}\frac{\sin^2(Nkd\sin(\theta)/2)}{\sin^2(kd\sin(\theta)/2)} \tag{66}$$

Suppose $a = d$ in the expression:

$$I = I_1 \frac{\sin^2(kd\sin(\theta)/2)}{(kd\sin(\theta)/2)^2}\frac{\sin^2(Nkd\sin(\theta)/2)}{\sin^2(kd\sin(\theta)/2)} \tag{67}$$

$$= I_1 \frac{\sin^2(Nkd\sin(\theta)/2)}{(kd\sin(\theta)/2)^2} \tag{68}$$

$$= I_1 N^2 \frac{\sin^2(k(Nd)\sin(\theta)/2)}{(k(Nd)\sin(\theta)/2)^2} \tag{69}$$

The result is a single slit with width Nd (When $a = d$ the spaces between the slits disappear).