1 Damped, driven oscillator, limiting cases [25 points]

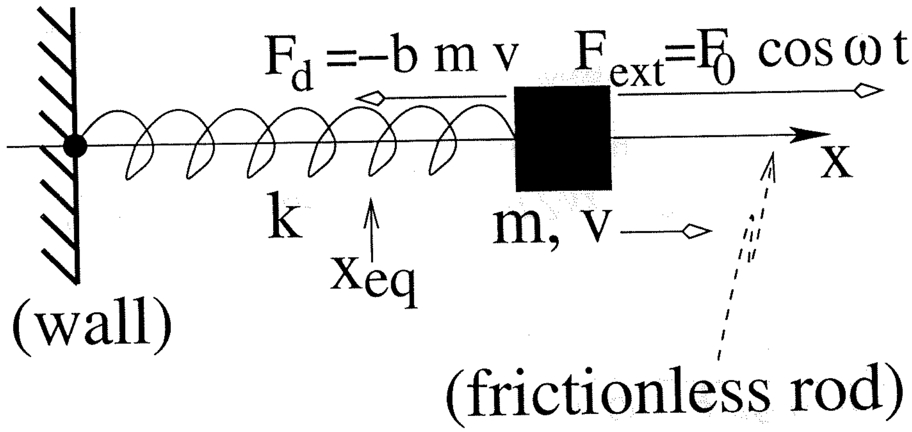


Figure 1: Damped, driven oscillator from lecture

Figure 1 shows the damped, driven oscillator studied in lecture. The system consists of a mass m which moves along frictionless rod along the x-axis under the influence of (1) a drag force $F_d = -bmv$, where v is the velocity of the mass; (2) a spring of spring constant k and equilibrium position x_{eq}; and (3) an external periodic drive force of amplitude F_0 with frequency ω and phase $\phi_0 = 0$, $F_{ext}(t) = F_0 \cos(\omega t)$.

In lecture, we showed that

$$x(t) = x_{eq} + \mathrm{Re}\left(\underline{A}e^{i\omega t}\right) \tag{1.1}$$

provided

$$\underline{A} = \frac{F_0/m}{\omega_0^2 - \omega^2 + ib\omega}.$$

where $\omega_0 \equiv \sqrt{k/m}$.

The problems starting on the following pages consider various simplifications of this system.

(a) Spring and external force only (12 points)

Consider the case where $m = 0$ and there is no drag ($b = 0$) so that the system consists basically of only a spring tied to the external force. Physics still determines the position of the endpoint of the spring. Note that, because there is so little left in this problem, some of the answers below are very simple; i.e., there is only one degree of freedom.

What is the degree of freedom (D of F) for this system?

$\boxed{x(t)}$ the position of the mass/location of end point of spring.

What is the equation of motion (E of M)?

$$\sum F_x = \cancel{m}^{0} a_x = 0$$

$$\boxed{-k(x - x_{eq}) + F_0 \cos\omega t = 0}$$

Using the complex representation, *find the value of* $\underline{A}$ so that Eq. (1.1) solves the equation of motion.

$$-k\left(\cancel{x_{eq}} + \mathrm{Re}(\underline{A} e^{i\omega t}) - \cancel{x_{eq}}\right) + F_0 \,\mathrm{Re}\, e^{i\omega t} = 0$$

$$\mathrm{Re}\left[(F_0 - k\underline{A}) e^{i\omega t}\right] = 0$$

$$\Rightarrow \quad \boxed{\underline{A} = \frac{F_0}{k}}$$

Is your solution above a *general solution*? Why or why not?

Yes. (1) It solves the E of M

(2) The E of M has zero time derivatives, thus we don't need any adjustable params & we don't have any.

(b) Mass and external force only (9 points)

Consider the case where there is no drag ($b = 0$) and no spring ($k = 0$), so that there is only a mass m and the external force.

Using the complex representation, *find the value of $\underline{A}$* so that Eq. (1.1) solves the equation of motion.

$$\sum F_x = m a_x$$

$$F_0 \cos\omega t = m \frac{d^2 x}{dt^2}$$

$$F_0 \, \mathrm{Re}\, e^{i\omega t} = m \frac{d^2}{dt^2}\left(\cancel{x_{eq}}^{0} + \mathrm{Re}\, \underline{A} e^{i\omega t}\right)$$

$$= \mathrm{Re}\left(-m\omega^2 \underline{A} e^{i\omega t}\right)$$

Thus, need $F_0 = -m\omega^2 \underline{A}$.

$$\Rightarrow \boxed{\underline{A} = \frac{-F_0}{m\omega^2}}$$

(c) Challenge: Limiting cases (4 points)

Each of the simplified problems

(a) External force and spring only ($m = 0$, $b = 0$)

(b) External force and mass only ($k = 0$, $b = 0$)

corresponds to one of the scenarios below for the complete problem (mass $m > 0$, spring constant $k > 0$, drag constant $b > 0$ and external force $F_0 > 0$).

Apply force very gradually (low frequency): $\omega << \omega_0$ and $\omega << b$	(a)
Apply force back and forth very rapidly (high frequency): $\omega >> \omega_0$ and $\omega >> b$	(b)

Put the letter of the appropriate simplified problem in the box next to the corresponding limit. **Hint:** You also can do this by physical reasoning rather than trying to solve the problem mathematically.

Gradual Application: mass moves slow, so drag is negligible.
Also, acceleration is tiny, so forces are in balance.
Rapid Application: mass can't keep up with force, so doesn't move much, so drag + spring are negligible.

OR (mathematically)

$$\lim_{\omega \to 0} \frac{F_0/m}{\omega_0^2 - \omega^2 + ib\omega} = \frac{F_0}{m\omega_0^2} = \frac{F_0}{m\left(\sqrt{\frac{k}{m}}\right)^2} = \frac{F_0}{k} \text{ (answer to (a)!)}$$

$$\lim_{\omega \to \infty} \frac{F_0/m}{\omega_0^2 - \omega^2 + ib\omega} = \frac{-F_0}{m\omega^2} \text{ (answer to (b)!)}$$

2 Lab Experiment I [25 points]

This problem considers a modification of the sound tube experiment from Lab I. The new experiment uses a tube of a *different length*, with two *closed* ends and filled with an *unknown* gas and with a modified speaker placed in the *center* of the tube. (See Figure 2.) This speaker consists of a

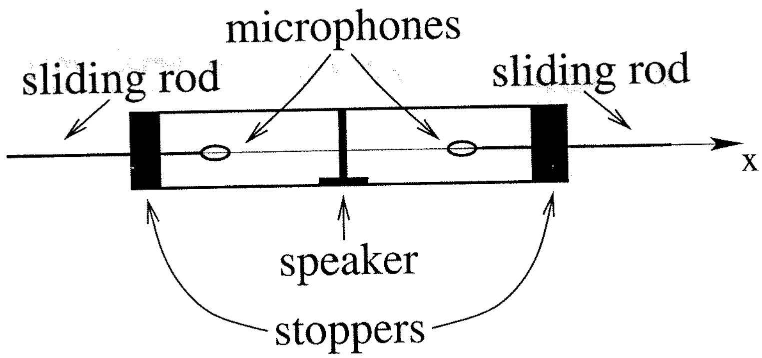


Figure 2: Modification of sound tube experiment from Lab I.

very thin solid membrane that moves back and forth, carrying the air on either side along with it. Finally, to measure the waves in the tube, two microphones sit upon a rods which slide through the stoppers at both ends to allow measurements at various positions.

At a frequency of $f_0 = 2000$ Hz, you find a clear resonance in the tube. For this frequency, you record the *amplitude* of the oscillations seen on the oscilloscope screen at a series of different positions in the tube. Figure 3 shows this amplitude, converted to pressure measured in newtons per square meter, plotted as a function of microphone position. Note that the graph shows no measured values at positions near the edges and near the speaker, locations which you cannot access due to the size of the microphone. Although some of the data are missing, you can assume that there is no space for any extra oscillations in the missing data.

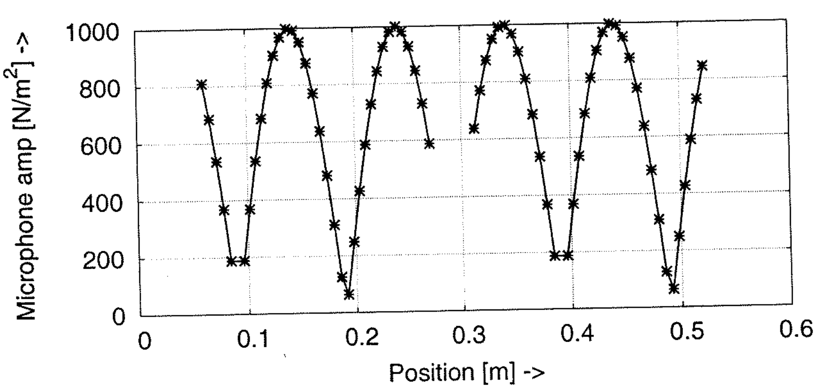


Figure 3: Data from modified sound tube experiment

(a) Speed of sound (10 points)

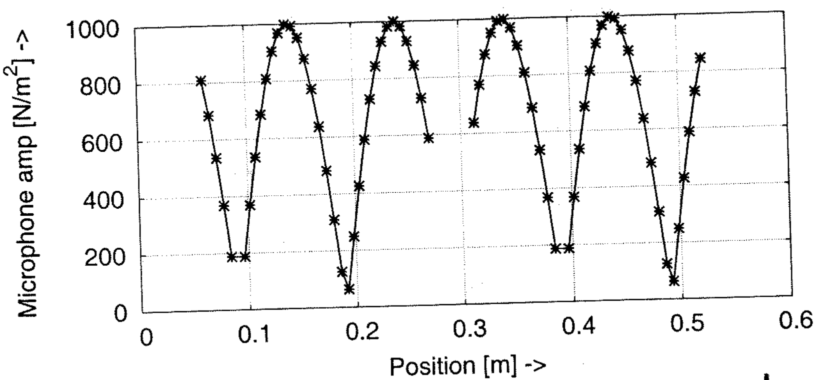
From the information given, determine an approximate value for the speed of sound of the *unknown* gas in the tube in units of m/s.

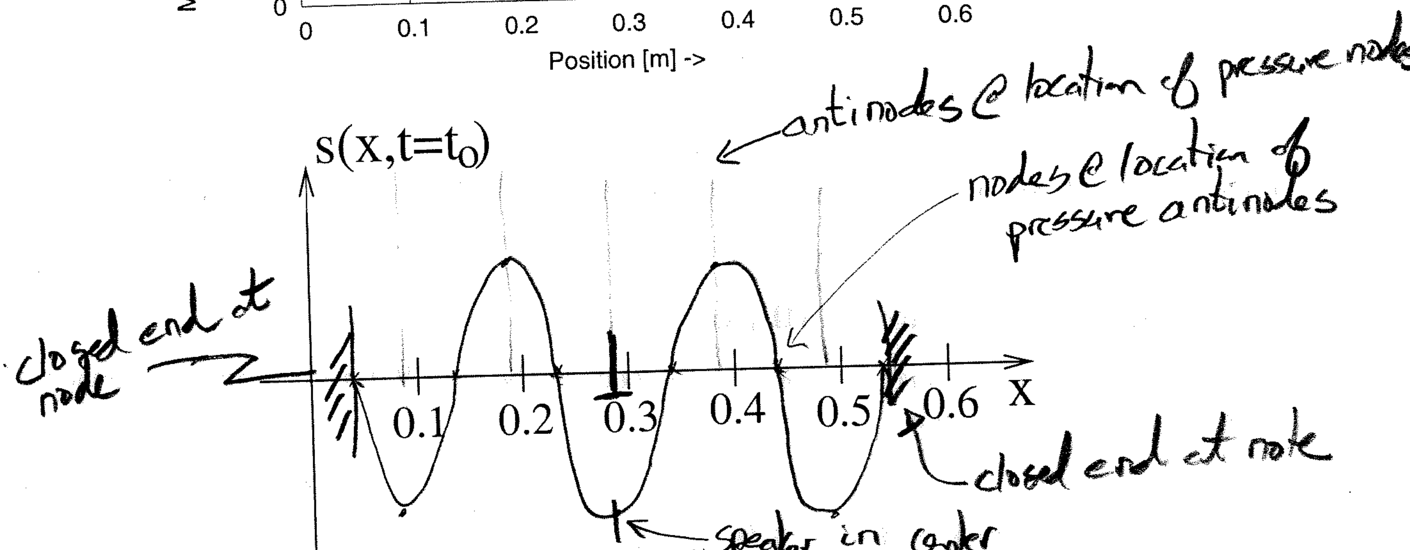
Spacing between nodes $= \frac{\lambda}{2} \approx 0.10\,m \Rightarrow \underline{\lambda \simeq 0.20\,m}$

$c = \lambda f_0 = (0.20\,m)(2000\,Hz) \approx \boxed{400\,m/s \approx c}$

(b) Displacement (10 points)

From the data in Fig. 3, *make a sketch on the axes below* of the shape of the corresponding gas displacement function $s(x,t)$ as a function of position x in the tube at a time $t = t_o$ for which $s(x,t) \neq 0$. *Indicate* on your sketch the locations of the ends of the tube *and* the location of the speaker. Be sure to *align your sketch* with the pressure pattern shown below.





[put sketch here]

(c) Length of the space in the tube (5 points)

Give, in units of m, an approximate numerical value of the length L of the space in the tube which is filled with the gas.

$$L = (2+\tfrac{1}{2})\lambda = \tfrac{5}{2}\lambda = 0.5 \text{ m}$$

3 Strings and springs [30 points]

Figure 4 shows a standard string system from lecture of length L, mass per unit length μ, and applied tension τ. But, now each end (rather than being fixed) is attached to a *massless* ring which slides along a frictionless rod and is attached to a spring with spring constant S and equilibrium position $y = 0$. Note that we use S for the *spring constant* to avoid possible confusion with the *wave-vector* k of waves in this system. Finally, the y-axis is horizontal and the tension τ far exceeds the weight of the string so that you may completely ignore the effects of gravity in this problem.

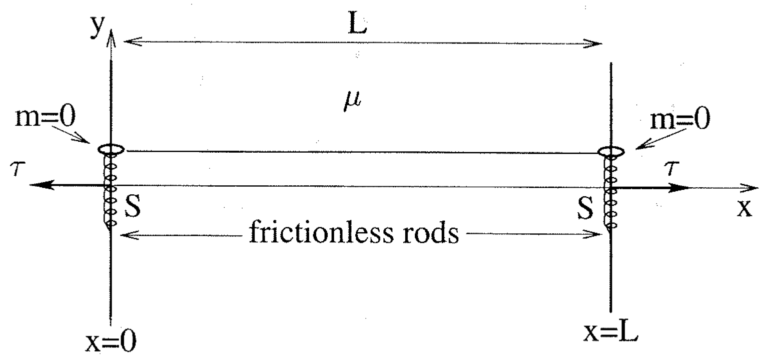


Figure 4: Standard string system but with modified free boundary conditions with springs

(a) Boundary conditions (10 points)

Derive the boundary conditions that apply to the ends of the string at $x = 0$ and $x = L$ when the string is not necessarily straight.

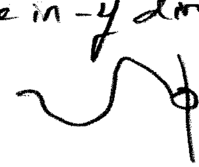
Boundary condition at $x = 0$:

+ slope gives force in +y direction:

$$\sum F = \cancel{m}^{0}\cdot a = 0$$
$$= \underbrace{-S\cdot y}_{\text{spring}} + \underbrace{\tau \frac{\partial y}{\partial x}}_{\text{tension}}$$
$$\Rightarrow \boxed{y(x=0,t) = \frac{\tau}{S}\frac{\partial y(x=0,t)}{\partial x}}$$

Boundary condition at $x = L$:

- slope gives force in -y direction:



$$\sum F = \cancel{m}^{0}\cdot a = 0$$
$$= -S\cdot y - \tau \frac{\partial y}{\partial x}$$
$$\Rightarrow \boxed{y(x=L,t) = -\frac{\tau}{S}\frac{\partial y(x=L,t)}{\partial x}}$$

Important note: If you were unable to do part **(a)**, assume for the rest of the problem that the boundary conditions are $\frac{\partial y}{\partial t}|_{x=0} = \frac{\tau}{S}\frac{\partial^2 y}{\partial t \partial x}|_{x=0}$ and $\frac{\partial y}{\partial t}|_{x=L} = -\frac{\tau}{S}\frac{\partial^2 y}{\partial t \partial x}|_{x=L}$. These are *not* the correct boundary conditions!

(b) Natural modes (10 points)

Derive two mathematical equations (one for $x = 0$ and one for $x = L$) in terms of no quantities other than ω, k, S, τ, μ and L which must hold in order that the natural mode solution $y(x,t) = A\cos(kx + \phi_0)\cos(\omega t)$ satisfy the boundary condition from **(a)**. (Do not attempt to solve them!)

Hints: You should be able to find simple expressions for the values of $\tan(\phi_0)$ and $\tan(kL + \phi_0)$. Remember that $\tan\theta = \sin\theta/\cos\theta$.

$x=0$:

$$y(x=0,t) = +\frac{\tau}{S}\frac{\partial y(x=0,t)}{\partial x}$$

$$\cancel{A}\cos(k\cdot\overset{0}{0}+\phi_0)\cancel{\cos\omega t} = -\frac{\tau}{S}k\cancel{A}\sin(k\cdot\overset{0}{0}+\phi_0)\cancel{\cos\omega t}$$

$$-\frac{S}{\tau k} = \frac{\sin\phi_0}{\cos\phi_0} = \boxed{\tan\phi_0 = \frac{-S}{\tau k}}$$

$x=L$:

$$y(x=L,t) = \frac{-\tau}{S}\frac{\partial y(x=L,t)}{\partial x}$$

$$\cancel{A}\cos(kL+\phi_0)\cancel{\cos\omega t} = +\frac{\tau}{S}k\cancel{A}\sin(k\cdot L+\phi_0)\cancel{\cos\omega t}$$

$$\frac{S}{k\tau} = \frac{\sin(kL+\phi_0)}{\cos(kL+\phi_0)} = \boxed{\tan(kL+\phi_0) = \frac{S}{k\tau}}$$

(c) Effective simple harmonic oscillator (5 points)

If the springs are relatively weak ($S << \tau/L$), in its lowest frequency natural mode, the string remains *almost* straight and oscillates up and down as a unit. Treating the system as a simple harmonic oscillator, estimate the angular frequency ω of this natural mode in terms of no quantities other than μ, τ, S, and L.

$$\omega = \sqrt{\frac{k}{m}} = \boxed{\sqrt{\frac{2S}{\mu L}} = \omega}$$

two springs, pulling together

total mass

(d) Challenge: Lowest mode (5 points)

When the spring is weak ($S << \tau/L$), the string in the lowest most is almost perfectly straight so that $kL << 1$. Under these conditions, you can use the small angle approximation $\tan\theta \approx \theta$ in *both* of your equations in **(b)**. After approximating both equations in this way, solve them and show that the angular frequency ω of the mode agrees exactly with your guess from **(c)**!!!

$$\tan\phi_0 = \frac{-S}{\tau k} \approx \phi_0$$

$$\tan(kL+\phi_0) = \frac{S}{\tau k} \approx kL + \phi_0$$

$$\frac{S}{\tau k} \approx kL - \frac{S}{\tau k}$$

$$\frac{2S}{\tau k} = kL$$

$$\frac{2S}{\tau L} = k^2$$

$$\Rightarrow \underline{k = \sqrt{\frac{2S}{\tau L}}}$$

$$\boxed{\omega = ck = \sqrt{\frac{\tau}{\mu}} \cdot \sqrt{\frac{2S}{\tau L}} = \sqrt{\frac{2S}{\mu L}}\ !}$$

4 Waves on a rotating rope [20 points]

Figure 4 shows a perspective and side view of a rope of mass density μ and length L rotating in the plane perpendicular to the y-axis at angular velocity Ω and undergoing vertical wave motion. Note that we use capital Ω for the angular *velocity* of the rotation to distinguish it from the angular *frequency* ω of waves in the system. You may take the rope as rotating fast enough that you may ignore gravity.

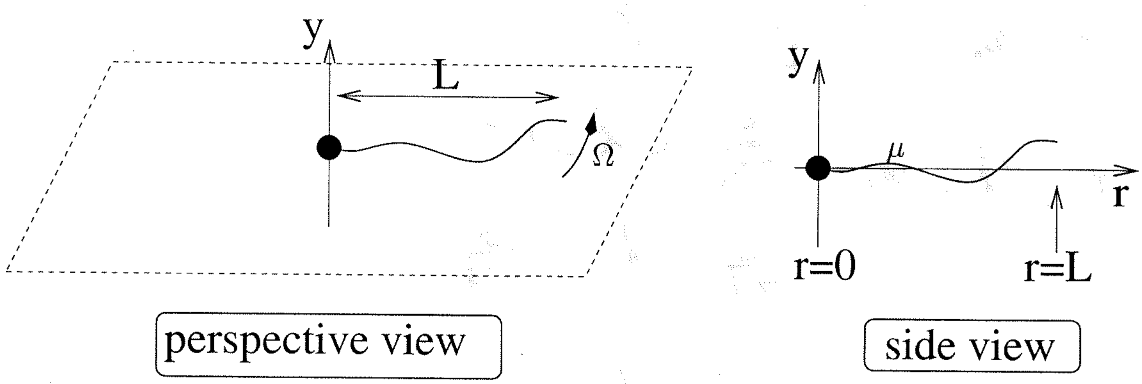


Figure 5: Analysis of vertical oscillations of a rotating rope.

Analogous to waves on a normal string, the function $y(r,t)$ giving the *vertical displacement* along y of the chunk which originated at distance r from the center of rotation makes up the appropriate degrees of freedom.

(a) Equation of motion (15 points)

In terms of $\tau(r)$ (defined as the radial component of the tension in the rope at distance r from the center), $y(r,t)$, and appropriate derivatives, what is $\mu\partial^2 y(r,t)/\partial t^2$?

Hint: To save time, you can start with the differential form of Newton's law for strings from the formula sheet and not bother with a free-body diagram.

$$\frac{\partial T_y(r,t)}{\partial r} = \mu\frac{\partial^2 y}{\partial t^2} \quad \text{(From Formula Sheet)}$$

θ : $T_y(r,t)$, $\tau(r)$

$$\frac{T_y(r,t)}{\tau(r)} = \tan\theta = \frac{\partial y}{\partial r} \Rightarrow T_y = \tau(r)\cdot\frac{\partial y}{\partial r}$$

$$\Rightarrow \boxed{\mu\frac{\partial^2 y(r,t)}{\partial t^2} = \frac{\partial}{\partial r}\left(\tau(r)\frac{\partial y(r,t)}{\partial r}\right)}$$

(b) Challenge: Standing wave (5 points)

For the lowest natural mode of this unusual system, the rope stays perfectly straight with its slope oscillating up and down, $y(r,t) = A\cos(\omega t)\cdot r$. In terms of no quantities other than μ, L and Ω, what is the angular *frequency* ω of these oscillations?

Hints:

- Use the result of applying Newton's law in the radial direction that $\tau(r) = \mu\Omega^2\left(L^2 - r^2\right)/2$.
- The correct answer has a very simple form.

$$\mu\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial r}\left[\frac{\mu\Omega^2}{2}(L^2-r^2)\frac{\partial y}{\partial r}\right]$$

$$-\mu\omega^2 A\cos(\omega t)\cdot r = \frac{\partial}{\partial r}\left[\frac{\mu\Omega^2}{2}(L^2-r^2)\cdot A\cos\omega t\right]$$

$$= \frac{\partial}{\partial r}\left[\cancel{\underbrace{\frac{\mu\Omega^2}{2}L^2 A\cos\omega t}_{\text{const}}} - \frac{\mu\Omega^2 A\cos\omega t}{2}r^2\right]$$

$$= -\mu\Omega^2 A\cos(\omega t)\cdot r$$

$$\cancel{\mu}\omega^2\cancel{A}\,\cancel{\cos(\omega t)\cdot r} = \cancel{\mu}\Omega^2\cancel{A}\,\cancel{\cos(\omega t)\cdot r}$$

$$\Rightarrow \boxed{\omega = \Omega}$$