1 Problem 1: Traveling Electromagnetic Wave [20 points]

(a) (10 points)

Write an expression for the $\vec{E}$-vector of an electromagnetic plane wave that has wavelength λ and is traveling in vacuum with speed c in the negative z-direction. At $t = 0$ and at $(x, y, z) = (0, 0, L)$, $\vec{E}$ is pointing in the positive x-direction and has its maximum value, E_0. Your expression should be in terms of E_0, λ, c, L, and mathematical constants.

$$\vec{E}(z,t) = E_0 \cos(k(z-L) + \omega t)\,\hat{x}$$

polarization

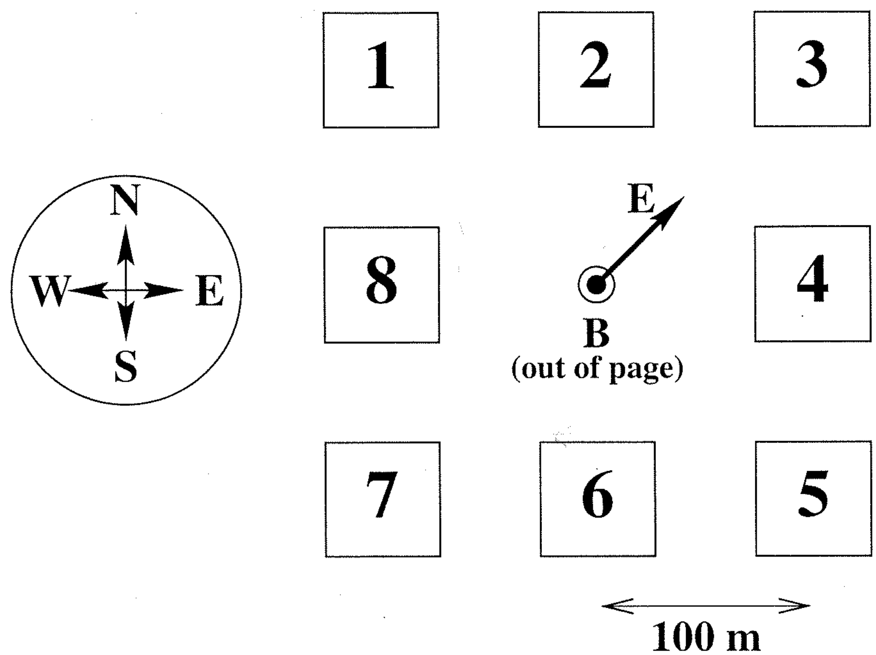
with $k = \frac{2\pi}{\lambda}$

$\omega = kc = \frac{2\pi}{\lambda} c$

$$\boxed{\vec{E}(z,t) = E_0 \cos\left[\frac{2\pi}{\lambda}(z - L + ct)\right]\hat{x}}$$

(b) (10 points)

At an unnamed Ivy League university, before the time of web sites and e-mail, a group of engineering students broadcast the solutions to the homework for their physics course, a day before they were due. The faculty knew that the transmitter must be in a building at the local engineering quad, and set up a receiver in the center of the quad which could detect the direction and strength of both the electric and the magnetic field vectors of the radio waves. The directions detected at a time t are shown in the picture on the next page.



(i) **(5 points)**
If the students broadcast from only one building, which one must it have been? Enter the number of the building in the box.

right hand rule
wave propagating from 1 → 5

$\boxed{1}$

(ii) **Challenge** (5 points)
One of the TA's noticed that the magnitude of the magnetic field vector was by a factor of $\sqrt{2}$ larger than expected from a single source, and concluded that there had to be two transmitters of equal power, transmitting in phase. Which two building numbers are consistent with this hypothesis?

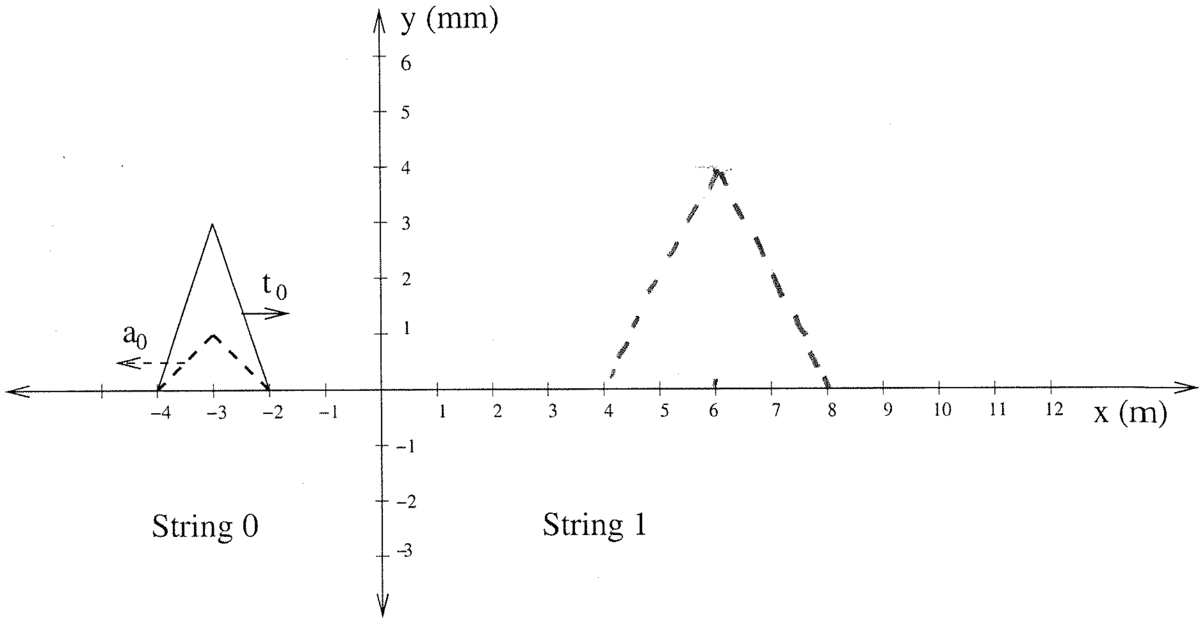
$\vec{E}$
2,4 $\rightarrow\uparrow$, $\vec{B}$ would be $\odot$ from 2, $\otimes$ from 4, $\vec{B}=0$
4,6 $\uparrow\rightarrow$ $\vec{B}$ would be $\otimes$ from 4, $\otimes$ from 6, $\vec{B}=2B_0$ down - wrong directio
6,8 $\rightarrow\uparrow$ $\vec{B}$ would be $\otimes$ from 6, $\odot$ from 8, $B=0$
8,2 $\uparrow\rightarrow$ $\vec{B}$ " " $\odot$ from 8, $\odot$ from 2, $B=2B_0$ up

$\boxed{2,\ 8}$

$\vec{E}=\sqrt{2}\,E_0$

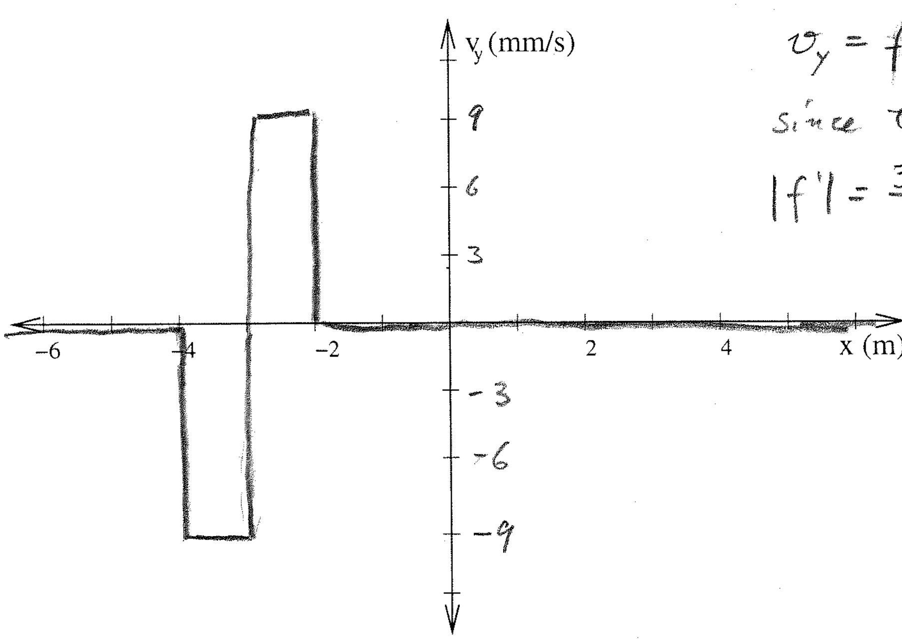
2 Problem 2: Pulses at a Boundary [30 points]

Two infinitely long strings, stretched in the x-direction with tension τ, are tied together at $x = 0$. The string on the left side ($x \leq 0$) has impedance Z_0. The figure below shows a pulse (labeled t_0) coming in from the left with speed $c_0 = 3$ m/s at a time $t = 0$ (solid line), and the resulting reflected pulse (labeled a_0), at time $T = 2$ s (dashed line).



(a) (10 points)

Using the coordinate system below, make a graph of the particle velocity v_y of the strings at time t as a function of x. Be sure to numerically label the vertical axis.



$v_y = f'(-c_0)$

since t_0 is $f(x-c_0t)$

$|f'| = \frac{3\,mm}{1\,m}$ $|v_y| = \frac{9\,mm}{s}$

(b) (5 points)

Show that the impedance of the string at the right is $Z_1 = \frac{1}{2}Z_0$.
Hint: You may use any formula that you recall from lecture, or from the formula sheet.

$$R_{0\to 1} = \frac{Z_0 - Z_1}{Z_0 + Z_1} = \frac{1}{3}$$

$$3Z_0 - 3Z_1 = Z_0 + Z_1$$

$$4Z_1 = 2Z_0$$

$$Z_1 = \frac{1}{2}Z_0$$

(c) (5 points)

Calculate the height of the transmitted pulse, in the units given in the figure (mm).

transmission coefficient $T_{0\to 1} = \frac{2Z_0}{Z_0 + Z_1}$

$$= \frac{2Z_0}{Z_0 + \frac{1}{2}Z_0}$$

$$= \frac{4}{3}$$

$\boxed{\text{height} = 4\text{ mm}}$

(d) (10 points)

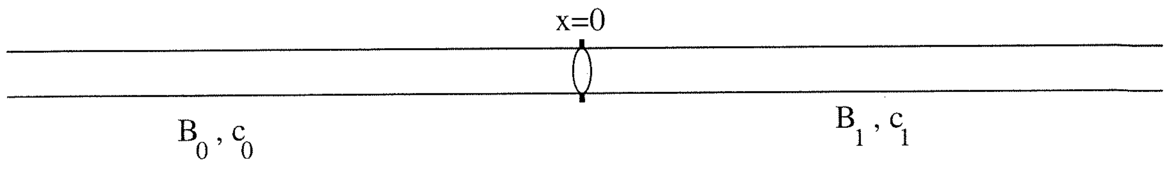
Calculate the speed of the transmitted pulse, and sketch the transmitted pulse at time $T = 2$ s in the same figure that shows the incoming and reflected pulses. Indicate clearly its height, position, and width.

$Z = \frac{T}{c}$ $\qquad \frac{c_1}{c_0} = \frac{Z_0}{Z_1} = 2$ $\qquad \boxed{c_1 = 2c_0 = 6\text{ m/s}}$

peak of pulse at $x = 6$ m
pulse widened by factor of 2

3 Problem 3: Impedance Matching by Damping [30 points]

Two infinitely long pipes of cross sectional area A extend in the x-direction and are connected at $x = 0$. The pipe on the left is filled with air of bulk modulus B_0, the one on the right with a different gas of bulk modulus B_1. The equilibrium pressure in both pipes is p_0. The speed of sound is c_0 and c_1, respectively, and $B_0/c_0 > B_1/c_1$.
A massless membrane, $m = 0$, is inserted at $x = 0$; it keeps the two gases separate and it provides a damping force $-bA(\partial s/\partial t)$.



A sinusoidal traveling wave of frequency ω and wave vector k_0 is coming in from the left. The solution to the wave equation has the form

$$s_0(x \le 0, t) = \Re\left[\underline{A}e^{-i\omega t}(e^{ik_0x} + \underline{R}e^{-ik_0x})\right] \tag{3.1}$$
$$s_1(x \ge 0, t) = \Re\left[\underline{A}e^{-i\omega t}\underline{T}e^{ik_1x})\right] \tag{3.2}$$

(a) (10 points)

Explain why the following two equations (boundary conditions) hold at $x = 0$. Your explanation should include an identification of the terms in Equations (3) and (4)

$$s_0(x=0,t) = s_1(x=0,t) \tag{3.3}$$

$$\left(B_1 \left.\frac{\partial s_1}{\partial x}\right|_{x=0} - B_0 \left.\frac{\partial s_0}{\partial x}\right|_{x=0} \right) - b\left.\frac{\partial s_1}{\partial t}\right|_{x=0} = 0 \tag{3.4}$$

(3.3) continuity at $x = 0$, displacement functions are equal

(3.4) force equilibrium $\sum F = ma$

$B_1 \frac{\partial s_1}{\partial x}$ force per unit area from Gas(1) on membrane

$-B_0 \frac{\partial s_0}{\partial x}$ " " " " " " (0) " "

$-b\frac{\partial s_1}{\partial t}$ friction force per unit area " "

$m = 0 \Rightarrow m \times$ accel. $= 0$ on right hand side

(b) (6 points)

Use the boundary conditions along with the form of the solution to the wave equation to find two equations for $\underline{R}$ and $\underline{T}$ in terms of B_0, B_1, b, k_0, k_1, c_0, c_1 and ω, as needed.

(3.3) $\underline{A}e^{-i\omega t}$ drops out, $e^{ikx}\big|_{x=0} = 1$

from (3.3) $1 + \underline{R} = \underline{T}$

from (3.4) $B_1 i k_1 \underline{T} - B_0 i k_0 (1 - \underline{R}) - b(-i\omega)\underline{T} = 0$

(c) (9 points)

Solve for $\underline{R}$ and $\underline{T}$ and express your answers in terms of the impedances Z_0 and Z_1 and the drag constant b. (Hint: use $k = \omega/c$).

$$B_1 k_1 (1 + \underline{R}) - B_0 k_0 + B_0 k_0 \underline{R} + b\omega \underline{R} + b\omega = 0$$

$k = \frac{\omega}{c}$ divide by ω

$$\frac{B_1}{c_1} + \frac{B_1}{c_1}\underline{R} - \frac{B_0}{c_0} + \frac{B_0}{c_0}\underline{R} + b\underline{R} + b = 0$$

$$Z_1 + Z_1\underline{R} - Z_0 + Z_0\underline{R} + b\underline{R} + b = 0$$

$$\underline{R}(Z_1 + Z_0 + b) = Z_0 - Z_1 - b$$

$$\underline{R} = \frac{Z_0 - (Z_1 + b)}{Z_0 + Z_1 + b}$$

$$\boxed{\underline{R} = \frac{Z_0 - (Z_1 + b)}{Z_0 + (Z_1 + b)}}$$

$$\underline{T} = 1 + \underline{R} = \frac{Z_0 + \cancel{Z_1} + \cancel{b} + Z_0 - \cancel{Z_1} - \cancel{b}}{Z_0 + Z_1 + b}$$

$$\boxed{\underline{T} = \frac{2Z_0}{Z_0 + (Z_1 + b)}}$$

(d) (2 points)

For a certain value of b the wave is entirely transmitted ($\underline{T} = 1$), and there is no reflected wave ($\underline{R} = 0$). The impedances of the two sides are "matched". Find this value of b in terms of the impedances Z_0 and Z_1, and other quantities as needed.

$$1 = \frac{2Z_0}{Z_0 + Z_1 + b}$$

$$Z_0 + Z_1 + b = 2Z_0$$

$$b = Z_0 - Z_1$$

$$\boxed{Z_0 - Z_1}$$

(e) (3 points)

Try to guess this one even if you did not finish Parts (b) and (c). Suppose you only have one very long pipe of impedance Z_0, open at $x = 0$, (i.e. $Z_1 = 0$). How could you match this open pipe to the outside, using a membrane with friction?

$Z_1 = 0$

but the end with the membrane should look like it has impedance Z_0

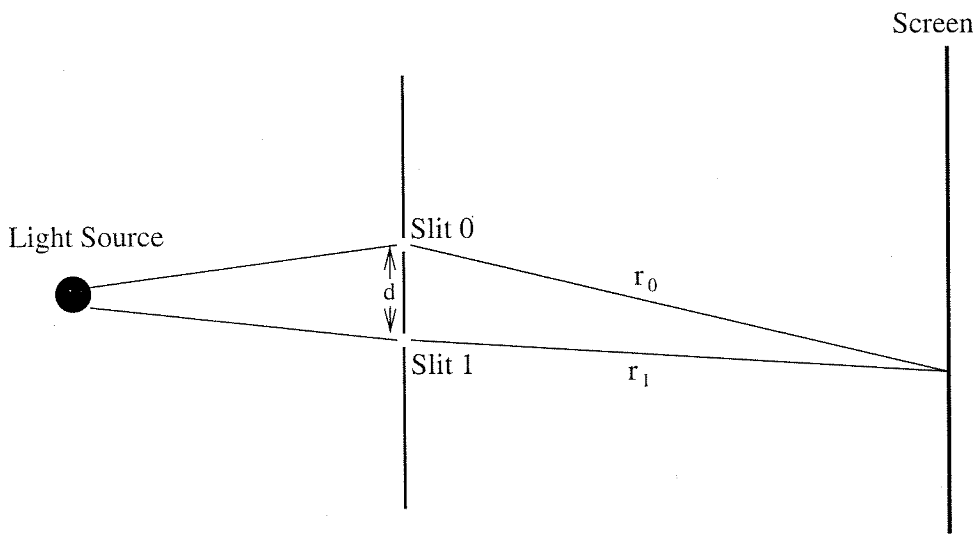
make $b = Z_0$

This gives $\underline{R} = 0$ using results from (c)

$\underline{T} = 1$

4 Problem 4: Double Slit Interference [20 points]

Two narrow parallel slits of width a_0 and a_1 are placed very far from a screen as shown in the figure. They are illuminated from the left with coherent light, such that the light waves emanating from the slits are in phase. The distances from the slits to a point on the screen are r_0 and r_1, respectively, where both r_0 and r_1 are very large compared to the distance d between the slits. The wavelength of the light is λ, with $\lambda << r_0$ and r_1. Assume that the intensity on the screen at distance r_0 due to one slit of width a_0 alone is I_0, and that the screen is far enough away from the slits that I_0 is essentially constant over the entire screen.



In terms of I_0, calculate the intensity for the following cases:

(a) (5 points)

$r_1 = r_0$
$a_1 = a_0$

$$I = |\sqrt{I_0}e^{ikr_0} + \sqrt{I_0}e^{ikr_0}|^2 = I_0|e^{ikr_0}|^2|1+1|^2$$

$$\boxed{4I_0}$$

(b) (5 points)

$r_1 = r_0 + \lambda/2$
$a_1 = a_0$

$$e^{ik(r_0+\frac{\lambda}{2})} = e^{i\frac{2\pi}{\lambda}(r_0+\frac{\lambda}{2})} = e^{ikr_0}\underbrace{e^{i\pi}}_{-1}$$

destructive interference

$$I = I_0 \left| e^{ikr_0} - e^{ikr_0} \right|^2$$

$$\boxed{0}$$

(c) (5 points)

$r_1 = r_0 + \lambda/4$
$a_1 = a_0$

$$I = I_0 \left| e^{ikr_0} \right|^2 \left| 1 + e^{i\frac{\pi}{2}} \right|^2 = I_0 \underbrace{\left| 1 + i \right|^2}_{(\sqrt{2})^2}$$

$$\boxed{2I_0}$$

(d) (5 points)

Challenge!

$r_1 = r_0$
$a_1 = 2a_0$

$$I = \left| \sqrt{I_0}\, e^{ikr_0} + \underbrace{\sqrt{I_0}\, e^{ikr_0} + \sqrt{I_0}\, e^{ikr_0}}_{\text{double-width slit acts like two right next to each other}} \right|^2$$

$$= 9I_0$$

$$\boxed{9I_0}$$