Physics 214: Waves, Optics, and Particles

Fall 2006

Homework Assignment # 2

(Due Thursday, September 7 at 5:00pm *sharp.*)

Agenda and readings for the week of September 4:

Skills to be mastered:

- differentiating between the Equation of Motion and a solution of it ;
- verifying a general solution;
- finding particular solutions given initial conditions;
- determining the complex amplitude from the initial conditions for a simple harmonic oscillator;
- determining the (real) amplitude and the initial phase from the complex amplitude;
- using the complex representation to solve differential equations.

Lectures and Readings (all optional except LN):

LN = the course lecture notes at http://people.ccmr.cornell.edu/∼muchomas/P214.
VW = A. P. French, *Vibrations and Waves.*
AG = the draft chapters by Alan Giambattista at http://www.physics.cornell.edu/p214.
YF = Young and Freedman, *University Physics*, 11th edition.

- Lec 4, 9/5 (Tue): Damped, driven oscillator; resonance.
  Readings: LN "Simple Harmonic Motion," Sec. 6; VW pp. 62-68 and 77-92); AG *Oscillations* Sections 24.6 and 24.7.
- Lec 5, 9/7 (Thu): Wave equation for the string; standing waves.
  Readings: LN "Intro to Waves: Waves on a String and Standing Waves," Sec. 1-4.3.1; VW pp. 161-167; AG *Waves* Sections 25.4 and 25.5; AG *Superposition* Sections 26.3 and 26.4.

1 Does it really work?

Take your answers for $x_{\text{eq}}, A, \omega_0$, and ϕ_0 from Problem 2.2b of Problem Set 1 and submit a plot of the function

$$x(t) = x_{\text{eq}} + A\cos(\omega_0 t + \phi_0), \tag{1}$$

checking to see that it agrees with Figure 1 of Problem Set 1.

2 Counting degrees of freedom and adjustable parameters

Consider a chain consisting of N point particles of mass m all connected by rubber bands, where the two end particles are held fixed and the remaining $N-2$ particles can move freely in the vertical direction but are otherwise constrained to move along vertical rods. (See Figure 1.) The rods are frictionless, but each particle experiences forces from the rubber bands, gravity, and air resistance of the form $F=-bv^2$, where v is the velocity of the particle.

(a) What are the degrees of freedom of this system?

(b) How many degrees of freedom are there?

(c) How many adjustable parameters will there be in the general solution to the equations of motion?

(d) How many adjustable parameters would there be in the general solution if the end particles are fixed but the other particles are *not* constrained by the rods?

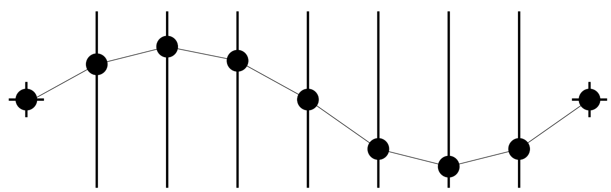

Figure 1: The chain described in Problem 2.

3 Identifying solutions and general solutions

The equation of motion (E of M) for an ideal mass-spring system is

$$-k(x-x_{\text{eq}})=m\frac{d^2x}{dt^2}$$

For each of the following expressions, answer these questions:

(i) Can it represent a solution to the E of M? If so, specify the required values of constants that are not adjustable parameters. If not, briefly explain why not.

(ii) If the answer to (i) is yes, can it represent a *general* solution to the E of M? If so, identify the adjustable parameters.

Note: All quantities other than x and t are constants. Assume all constants other than i to be real. In some cases, two or more different symbols (*e.g.* C_2 and C_5) may end up standing for the same constant. The symbol $\Re\{\}$ means *the real part of.*

(a) $x(t)=\Re\{C_1+(C_2+C_3i)\cos C_4t\}$

(b) $x(t)=\Re\{C_1+C_2e^{iC_3t}+C_4e^{iC_5t}\}$

(c) $x(t)=x_{\text{eq}}+A\sin\left[\omega_0(t-t_0)\right]$

(d) $x(t)=x_{\text{eq}}+(A\sin\omega_1 t)(B\sin\omega_2 t)$

(e) $x(t)=x_{\text{eq}}+A\sin\omega_0(t-t_0)$

4 Oscillator with two kinds of damping

An oscillator is modeled as a mass m at equilibrium point $x_{\text{eq}} = 0$ that slides on a horizontal surface along the x axis. The mass is acted on by three horizontal forces: (1) the force due to an ideal spring of spring constant k; (2) a sliding frictional force with magnitude

$$|f_{\text{f}}| = \mu m g$$

and (3) a viscous damping force given by

$$f_d = -bm\frac{dx}{dt}$$

(a) Starting with $\sum \vec{F} = m\vec{a}$, derive the equation of motion that applies when the mass is moving *to the left* (in the $-x$ direction).

(b) The general solution to the equation of motion derived in (a) has the form

$$x(t) = C + \Re\{\underline{A}e^{i\underline{\omega}t}\}$$

where $\underline{A}$ and $\underline{\omega}$ are complex. Find the general solution by finding the values of C and $\underline{\omega}$ that make this solve the equation of motion. (We then know it is the general solution because we have two adjustable parameters, the real and imaginary parts of $\underline{A}$.)

Hints:

Do *not* assume that $C = x_{\text{eq}}$. You will find that a nonzero value of C is required to make our form for $x(t)$ solve the equation of motion.

Assume the form given for $x(t)$, take its time derivatives, and plug them all into the equation of motion. Now determine the values of $\underline{\omega}$ and C that are required to satisfy the equation of motion at *all* t. You don't have to explicitly take the real part (yet).

Use this theorem: If $\Re\{\underline{B}e^{i\underline{\omega}t}\} = 0$ for all t, where $\underline{B}$ is some complex expression that does not depend on time and $\Re\{\underline{\omega}\} \neq 0$, then $\underline{B}$ must equal zero. (You will prove this on the next assignment.)

Remember that if some complex expression is equal to zero, then both its real and imaginary parts must equal zero.

You will find two valid values for ω. Either one works just as well as the other. Pick one and work with it consistently. Also, to save some repetitious writing, define the two quantities

$$\omega_0 \equiv \sqrt{k/m}$$

and

$$\omega' \equiv \sqrt{\omega_0^2 - b^2/4}$$

and use these quantities in your answers.

If you get stuck, the case $\mu = 0$ (viscous damping but no sliding friction) is the standard damped oscillator that is solved in most textbooks, although not necessarily using the complex representation. The case $b = 0$ (sliding friction but no viscous damping) was done on PS2, Fall 2005. You can check your answer by plugging in $\mu = 0$ or $b = 0$ and seeing if your expression reduces to the correct solution for each of those two special cases.

(c) Suppose the mass is released from rest at $t = 0$ at position $x = x_0$, where $x_0 > 0$ and is sufficiently large that the mass begin to move once it is released. Find the particular solution that describes its motion from $t = 0$ until it (instantaneously) comes to rest. (In other words, find the value of the complex amplitude $\underline{A}$ for these initial conditions.)

(d) To check your answer to part (c), take the real part to show that

$$x(t) = \frac{\mu g}{\omega_0^2} + \left(x_0 - \frac{\mu g}{\omega_0^2}\right) e^{-bt/2} \left(\cos\omega' t + \frac{b}{2\omega'} \sin\omega' t\right)$$

Note that although you had to do some manipulations with complex numbers that might be new to you, you didn't have to do any tricky trig to get this complicated answer!

5 The meaning of phase

As mentioned in lecture, the "phase" of an oscillator can be thought of in analogy to the phases of the moon. Imagine a coordinate system set up so that the x-axis always points from the earth toward the sun with the earth sitting at $x = 0$. (See Figure 2.) In a very simplified view of the orbit of the moon, we take the orbit of the moon to be a circle of fixed radius $R = 384{,}467$ km and period $T = 29.5306$ days). If t_n is the exact time of a new moon, then the $x-$coordinate of the position of moon is given by

$$x(t) = R\cos\frac{2\pi(t - t_n)}{T}$$

ignoring the motion of the earth around the sun, the tilt of the moon's orbit, etc.

(a) According to the definition of phase given in class, what is the phase of the moon (in radians) when the moon is at first quarter, full, second quarter, and new?

(b) The moon was at its last quarter this past Aug 15 at 9:51 pm EDT, what will its phase be on the due date of this problem set, September 7 at 5:00 pm EDT, both in radians and in terms of new, full, etc.?

(c) If we instead set $t = 0$ to correspond to noon on the first day of classes, Aug 24, what is the initial phase of the moon (in radians)?

(d) If the initial phase had been $\pi/8$—that is, if at $t = 0$ the phase of the moon is $\pi/8$—for what value of t (in days) would the first full moon occur?

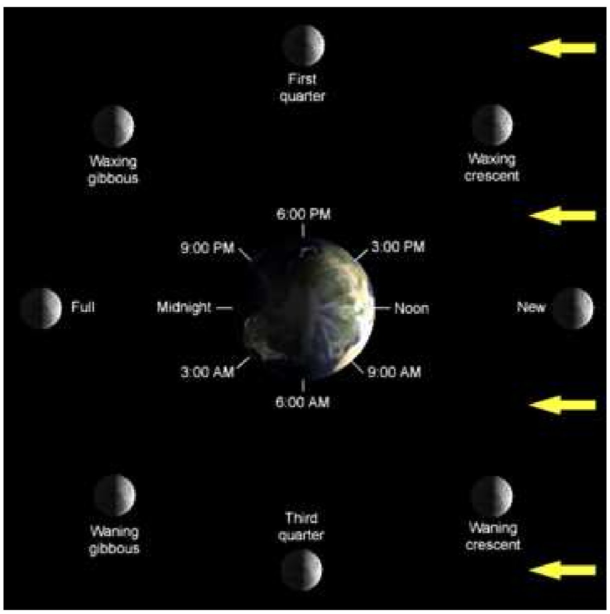

Figure 2: The moon's orbit around the earth. Earth is at the origin of the coordinate system and the x-axis points to the right. The sun (not shown) lies on the $+x$-axis. The moon moves counterclockwise in a circle around the earth. The yellow arrows represent sunlight.